

MATHEMATICAL ANALYSIS STUDIES

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1. NUMBERS, SETS AND FUNCTIONS

We start with studying number systems, some "naive set theory", and then define functions.

1.1. Natural Numbers. The naturals $\mathbb{N} := \{1, 2, \dots\}$ is axiomatically defined via Peano axioms, which we briefly mention them for completeness.

1.1.1. Peano axioms. There exists a non-empty set $\mathcal{N}$ and an injective function $S : \mathcal{N} \rightarrow \mathcal{N}$ (the successor of function) such that:

- (1). $1 \in \mathcal{N}$ and there is no $n \in \mathcal{N}$ such that $s(n) = 1$
- (2). If $S \subseteq \mathcal{N}$ such that $1 \in S$ and if some $n \in S$, we have $s(n) \in S$ then $S = \mathcal{N}$ (induction axiom)

Generally (2) above is used by considering a property pertaining a natural n that is: if $P(1)$ is true and if for certain $n_0 \in \mathbb{N}$, $P(n_0)$ is true $\implies P(n_0 + 1)$ is true, then that property $P(n)$ is true for all naturals n . We can also define two operations (functions) inductively as follows:

(i) **Sum/ Addition**

Let $+: \mathcal{N} \times \mathcal{N} \rightarrow \mathcal{N}$ be given by the formula below

$$+(m, n) = m + n := \begin{cases} m + 1 = s(m) & \text{if } n = 1 \\ m + s(n) := s(m + n) & \text{if } n > 1 \end{cases}$$

(ii) **Product/ Multiplication**

Consider the function $\cdot : \mathcal{N} \times \mathcal{N} \rightarrow \mathcal{N}$ given by

$$\cdot(m, n) = m \cdot n := \begin{cases} m \cdot 1 = m & \text{if } n = 1 \\ m \cdot s(n) = m \cdot n + m & \text{if } n > 1 \end{cases}$$

And with the "order relation" $<$ given by: $m < n$, for $m, n \in \mathcal{N}$ then there exists a $p \in \mathcal{N}$ such that $n = m + p$.

With these one obtains all known properties of naturals regarding the sum, multiplication and the order relation $<$. It must be stated that all properties are proven via the induction principle. From now onwards we denote the natural numbers by $\mathbb{N}$ instead of $\mathcal{N}$.

It is important to mention that what was given above is just a sketch of Peano's theory of naturals. For a more complete discussion refer to [Analysis I](#) by Terence Tao (Chapter 1)

1.2. Integers or Integral Numbers. The integers were invented to deal with "the problem of debt" and for our purposes we briefly mention that they are constructed from $\mathcal{N} = \mathbb{N}$ by an equivalence relation on $\mathcal{N}$, which we won't mention here but can be found at chapter 2 of Tao's Analysis 1 book. And after all formalities we obtain the set

$$\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$$

1.3. Rational Numbers. The rationals is the next number system and it was invented to deal with "the problem of dividing things equally". But we only need to know it is formally constructed from integers by another equivalence relation on $\mathbb{Z}$ and after this rigorous construction we can introduce the set of rationals as

$$\mathbb{Q} = \left\{ \frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N} \right\}$$

With it in mind we can proceed to the next topic: consequences of the induction axiom

Since we assumed induction as an axiom, we mention one of its immediate consequences the well-ordering principle in $\mathbb{N}$ which states that

Theorem 1.1. "every non-empty subset of $\mathbb{N}$ contains a smallest element."

Proof Let S be only non-empty subset of $\mathbb{N}$. We want to prove it contains a smallest element ($x \in S$ such that $x \leq \Delta$ for all $\Delta \in S$). Suppose there is no smallest element in S . Notice that $1 \notin S$ because $1 \in S$ is the smallest natural number (Why?). Then consider the following property pertaining a natural number n $P(n)$: for every $k \in \mathbb{N}$ such that $k \leq n$, $k \notin S$. Then we see that $P(n)$ is a property pertaining naturals S for which: $P(1)$ is true. And if $P(n)$ is true, meaning every $k \in \mathbb{N}$ such that $k \leq n_0$, we have that $k \notin S$. If $P(n_0 + 1)$ was false notice that there exists some $\tilde{k} = n_0 + 1$ and $\tilde{k} \in S$; but we know from the induction step (the assumption that $P(n_0)$ is true) that no $k \leq n_0$ belongs to S (all $k \leq n_0, k \notin S$), then $\tilde{k} = n_0 + 1$ and $n_0 + 1 \in S$

But this is impossible because $n_0 + 1$ would be the smallest natural of S , hence we reached a contradiction. Therefore there is a smallest element in S .

Problem 1.1. Using the well-ordering of $\mathbb{N}$, prove the following statements:

- (1) Prove that for all natural numbers n , if $n \geq 2$, then n is either prime or the product of prime numbers. Recall that a *prime number* is a natural number p greater than 1, whose only divisors are 1 and p itself.
- (2) Let $n \in \mathbb{N}$, a *divisor* of n is any natural number d such that there exists an integer k with $n = d \cdot k$. Consider the set $D_n = \{d \in \mathbb{N} : d \mid n\}$ of divisors of n . Prove that D_n has a least element.

2. ORDERED FIELDS

Definition 2.1. (Abstract Fields).

A field consists of a non-empty set $\mathbb{F}$ (called the set of $\mathbb{F}$ -numbers) and two operations (functions) $+: \mathbb{F} \times \mathbb{F} \longrightarrow \mathbb{F}$ and $\cdot: \mathbb{F} \times \mathbb{F} \longrightarrow \mathbb{F}$ called addition/sum and multiplication/product, with the following properties:

Axioms of Addition/Sum

- A.1.** For every $x, y \in \mathbb{F}$, we have $+(x, y) = +(y, x)$.
- A.2.** For every $x, y, z \in \mathbb{F}$, we have $+(x, +(y, z)) = +(+(x, y), z)$.
- A.3.** For every $x \in \mathbb{F}$, there exists $0 \in \mathbb{F}$ such that $+(x, 0) = +(0, x) = x$.
- A.4.** For every $x \in \mathbb{F}$, there exists $-x \in \mathbb{F}$ such that $+(x, -x) = +(-x, x) = 0$.

Axioms of Multiplication/Product

- M.1.** For every $x, y \in \mathbb{F}$, we have $\cdot(x, y) = \cdot(y, x)$.
- M.2.** For every $x, y, z \in \mathbb{F}$, we have $\cdot(x, \cdot(y, z)) = \cdot(\cdot(x, y), z)$.
- M.3.** For every $x, y, z \in \mathbb{F}$, we have $\cdot(x, +(y, z)) = +(\cdot(x, y), \cdot(x, z))$.
- M.4.** For every $x \in \mathbb{F}$, there exists $1 \in \mathbb{F} - \{0\}$ such that $\cdot(x, 1) = \cdot(1, x) = x$.
- M.5.** For every $x \in \mathbb{F} - \{0\}$, there exists $x^{-1} \in \mathbb{F}$ such that $\cdot(x, x^{-1}) = \cdot(x^{-1}, x) = 1$.

Conventions: We use the following conventions for shorthand notations. For every $x, y \in \mathbb{F}$, we denote $+(x, y)$ by $x + y$ and $\cdot(x, y)$ by $x \cdot y$. Additionally, for subtraction and division, we define $x - y := +(x, (-y))$ and $x/y := \cdot(x, y^{-1})$.

Problem 2.1. Show that additive inverse and multiplicative inverse of $x \in \mathbb{F}$ are unique.

Solution 2.1. Suppose on the contrary that $x \in \mathbb{F}$ has 2 additive inverses, this means $+(x, y) = 0 = +(x, z)$. From the equality of $+(x, y) = +(x, z)$, we get, $y = +(0, (-x))$ and $z = +(0, (-x))$. As *RHSs* are equal then *LHSs* are also equal: $y = z$.

By the same approach we get $y = \cdot(1, x^{-1})$, $z = \cdot(1, x^{-1})$ and then, $y = z$, proving uniqueness.

Problem 2.2. Prove that the $0 \in \mathbb{F}$ and $1 \in \mathbb{F}$ are unique

Solution 2.2. 1. Assume that there is another additive identity $0' \in \mathbb{F}$. By using additive identity property of 0 (from A3) $\longrightarrow +(0', 0) = 0'$. Then, by using additive identity property of $0'$ (from A3) $\longrightarrow +(0, 0') = 0$. Then from commutativity (A1), we have $+(0', 0) = +(0, 0')$, therefore $0' = 0$.

2. Assume that there is another multiplicative identity $1' \in \mathbb{F}$. By using multiplicative identity property of 1 (from M4) $\longrightarrow \cdot(1', 1) = 1'$. Then, by using multiplicative identity property of $1'$ (from M4) $\longrightarrow \cdot(1, 1') = 1$. Then from commutativity (M1), we have $\cdot(1', 1) = \cdot(1, 1')$, therefore $1' = 1$.

Problem 2.3. Prove that every linear equation $a \cdot x + b = 0$ on a field has a unique solution.

Solution 2.3. Assume by contradiction that there are two distinct solutions x_1 and x_2 for the equation $a \cdot x + b = 0$.

$$+(a, x_1) + b = 0 \quad \text{and} \quad +(a, x_2) + b = 0.$$

Subtracting the two equations, we get:

$$(+ (a, x_1) + b) - (+ (a, x_2) + b) = 0.$$

Simplifying the equation:

$$+(a, x_1) - +(a, x_2) = 0.$$

Therefore:

$$+(a, x_1) = +(a, x_2).$$

Using the inverse operation of addition, we subtract $+(a, x_2)$ from both sides:

$$x_1 = x_2.$$

This contradicts the assumption that $x_1 \neq x_2$. Therefore, our assumption that there are two distinct solutions is false.

Thus, the linear equation $a \cdot x + b = 0$ has a unique solution, which is given by:

$$x = \cdot(a^{-1}, -b).$$

Problem 2.4. Prove that $\cdot(0, x) = 0$ for every $x \in \mathbb{F}$

Solution 2.4. $\cdot(0, x) = \cdot(1 + (-1), x) = +(x, -x) \stackrel{A4}{=} 0$

Problem 2.5. Prove that $\cdot((-1), x) = -x$ for every $x \in \mathbb{F}$

Solution 2.5. $\cdot(1 + (-1), x) = +(\cdot(1, x), \cdot((-1), x)) = +(x, \cdot((-1), x)) = 0$ We know that additive inverse of x is $-x$, then from equality we will have $\cdot((-1), x) = -x$

Problem 2.6. Prove that $\cdot((-x), (-x)) = \cdot(x, x)$ for every $x \in \mathbb{F}$

Solution 2.6.

$$\begin{aligned} \cdot((-x), (-x)) &= \cdot(\cdot((-1), x), \cdot((-1), x)) = \cdot((-1), \cdot(x, \cdot((-1), x))) = \cdot((-1), \cdot(\cdot(x, (-1)), x)) = \\ &= \cdot((-1), \cdot((-1), \cdot(x, x))) = \cdot(\cdot((-1), (-1), \cdot(x, x))) \end{aligned}$$

By problem 2.4 we know that $\cdot((-1), x) = -x$ for every $x \in \mathbb{F}$. Since $1 \in \mathbb{F}$, $-1 \in \mathbb{F}$, then choosing $x = (-1)$ we find that $\cdot((-1), (-1)) = -(-1) = 1$ because $+\cdot((-1), -(-1)) = 0$ and there is only one element $x \in \mathbb{F}$ such that $+\cdot((-1), x) = 0$, which is $x = 1$. Hence, $-(-1) = 1$ and this shows that $\cdot((-x), (-x)) = \cdot(x, x)$ for every $x \in \mathbb{F}$

Definition 2.2. (Integer exponentiation in Fields).

For every $x \in \mathbb{F}, n \in \mathbb{N}$ we define x^n inductively by the following recursive scheme:

$$\begin{cases} x^1 & := x; \\ x^{n+1} & := x^n \cdot x \end{cases}$$

Problem 2.7. Prove that for every $m, n \in \mathbb{N}$ the following properties hold

- $x^{m+n} = x^{n+m}$
- $x^{m \cdot n} = x^{n \cdot m}$
- $x^{m \cdot (n+z)} = x^{m \cdot n} \cdot x^{m \cdot z}$

Solution 2.7. • We begin our induction by considering base case $n = 1$, $x^{n+1} = x^n \cdot x = x \cdot x^n = x^{1+n}$. Suppose that statement holds for all m_0, n_0 such that $1 \leq m_0, n_0 \leq m, n$. We are going to prove that $x^{m+(n+1)} = x^{(n+1)+m}$. By induction statement we already know that $x^{m+n} = x^{n+m}$, now if we multiply both sides with x , we get $x^{m+n} \cdot x^1 = x^{m+n+1} = x^{n+m+1} = x^{n+m} \cdot x^1$, so by induction $x^{m+n} = x^{n+m}$.

- For $n = 1$ we have, $x^{m \cdot 1} = x^m = x^m = x^{1 \cdot m}$. Assume that statement holds for m_0, n_0 such that $1 \leq m_0, n_0 \leq m, n$. We prove that $x^{m \cdot (n+1)} = x^{(n+1) \cdot m}$, but we already proven that $x^{m+n} = x^{n+m}$, and here we have $x^{m \cdot (n+1)} = x^{m \cdot n + m} = x^{m \cdot n} \cdot x^m = x^{n \cdot m} \cdot x^m = x^{(n+1) \cdot m}$.

- Consider base case $m = 1$, meaning $x^{m \cdot (n+z)} = x^{n+z}$, we also have $x^{m \cdot n} = x^n$ and $x^{m \cdot z} = x^z$, proving equality for the base case. Inductively we are going to prove that $x^{(m+1) \cdot (n+z)} = x^{(m+1) \cdot n} \cdot x^{(m+1) \cdot z}$. We already have, $x^{m \cdot (n+z)} = x^{m \cdot n} \cdot x^{m \cdot z}$, by multiplying both sides by $x^n \cdot x^z$, we get $x^{m \cdot (n+z)} \cdot x^n \cdot x^z = (x^{m \cdot n} \cdot x^n) \cdot (x^{m \cdot z} \cdot x^z)$ here we get, $x^{m \cdot (n+z) + 1 \cdot (n+z)} = x^{(m+1) \cdot (n+z)} = x^{(m+1) \cdot n} \cdot x^{(m+1) \cdot z}$.

Now we begin the study of ordered fields

Definition 2.3. Let $\mathbb{F}$ be a field. We say that $\mathbb{F}$ is an ordered field if, and only if, there exists a non-empty subset F^+ , called set of positive $\mathbb{F}$ -numbers, of $\mathbb{F}$ with the following properties

- O.1.** For every $a, b \in F^+$ we have $+(a, b) = a + b \in F^+$ and $\cdot(a, b) = a \cdot b \in F^+$
- O.2.** Given only $x \in \mathbb{F}$ exactly one of the following alternatives held
 - (a) $x \in F^+$;
 - (b) $-x \in F^+$;
 - (c) $x = 0$;

Then we say that $a < b$, here $a, b \in \mathbb{F}^+$, if and only if, $b + (-a) = +(b, (-a)) \in F^+$ ($a < b$ is used "a is less than b"). We also introduce the notation that $a \leq b$ to mean that either $a < b$, or $a = b$. The set $F^+ \cup \{0\}$ is called set of non-negative $\mathbb{F}$ -numbers, denoted F_0^+ . The set $F - F_0^+$ is the set of negative $\mathbb{F}$ -numbers, denoted F_0^- . $F_0^- \cup \{0\}$ is the set of non-positive $\mathbb{F}$ -numbers, denoted by F_0^- .

Problem 2.8. (a) Show that if $x > 0$, then $-x < 0$. Likewise, if $x < 0$, then $-x > 0$.

(b) Show that if $x > 0$, then $x^{-1} > 0$. Show that if $x > 1$, then $x^{-1} < 1$.

Solution 2.8. (a) $x > 0$, $+(x, (-x)) > +(0, (-x)) \rightarrow 0 > -x$, and if $x < 0$, $+(x, (-x)) < +(0, (-x)) \rightarrow 0 < -x$ (b) $x > 0$, $\cdot(x, x^{-1}) > 0$, as we have $x > 0$, then automatically $x^{-1} > 0$. If we have $x > 1$, by multiplying each side by x^{-1} we get $x^{-1} < 1$.

Problem 2.9. Prove the following "algebra" rules for inequalities in a field

- (1) if $a < b$ and $c \in \mathbb{F}$ then $a + c < b + c$;
- (2) if $a < b$ and $c \in \mathbb{F}^+$ then $a \cdot c < b \cdot c$;
- (3) if $a < b$ and $b < c$ then $a < c$;
- (4) Given any $a, b \in \mathbb{F}$ exactly one of the following statements hold:
 - (i). $a < b$; (ii). $b < a$; (iii). $b = a$

Solution 2.9. (1) $a, b \in \mathbb{F}^+, a < b \iff +(b, (-a)) \in F^+$, we can have $a + c < b + c$ if and only if we have $+(b + c, -(a + c)) \in F^+$

$$+(b + c, -(a + c)) = +(+(b, (-a)), +(c, (-c))) = +(b, (-a))$$

and we already know that $+(b, (-a)) \in F^+ \xrightarrow{\text{so}} +(b + c, -(a + c)) \in F^+$ also;

- (2) $a, b \in \mathbb{F}^+, a < b \iff +(b, (-a)) \in F^+; c \in \mathbb{F}^+, a \cdot c < b \cdot c \iff +(bc, -(ac)) \in F \iff \cdot(c, +(b, (-a))) \in F^+$. As we have $+(b, (-a)) \in F^+, c \in \mathbb{F}^+$, their product will also be in F^+ ;
- (3)

$$\begin{cases} a < b & \Rightarrow +(b, (-a)) \in F^+ \\ b < c & \Rightarrow +(c, (-b)) \in F^+ \end{cases}$$

Then their sum will also be in F^+ :

$$+(+(b, (-a)), +(c, (-b))) = b - a + c - b = c - a$$

So we have $+(c, (-a)) \in F^+$ and this means $a < c$;

(4) We will present 2 solutions for this part of the problem:

Solution 1. Suppose on the contrary that 2 or all of them hold, then we have

(a)

$$\left. \begin{array}{l} a < b \Rightarrow +(b, (-a)) \in \mathbb{F}^+ \\ b < a \Rightarrow +(a, (-b)) \in \mathbb{F}^+ \end{array} \right\} \Rightarrow 0 \in \mathbb{F}^+ \text{ (Contradiction)}$$

(b) Now suppose $a < b$ and $a = b$ hold, (from symmetry bja and a=b will be the same).

$$a < b \rightarrow +(b, (-a)) \in \mathbb{F}^+ \xrightarrow{a=b} 0 \in \mathbb{F}^+ \text{ Contradiction}$$

(c) Now suppose all hold. It is evident that from $a = b$ case we will again have $0 \in \mathbb{F}^+$, so contradiction in this case also.

We got contradiction for all assumptions, then we have only one of them to hold.

Solution 2. We know that $+(b, (-a)) \in \mathbb{F}$ for any $a, b \in \mathbb{F}$. And we have probability of its sign to be negative, positive or rather to be equal to 0 in equality case. And these all consequently mean: $a < b$ for positive, $b < a$ for negative, and $a = b$ for 0.

We can introduce the notion of intervals and bounded sets in any ordered field. In analogy with intervals studied in other subjects of high school mathematics we can introduce a notion of intervals in any abstract ordered field. For that let $a, b \in \mathbb{F}$ then we define

- $[a, b]_F := \{x \in \mathbb{F} : a \leq x \text{ and } x \leq b\}$, which is the closed interval (in $\mathbb{F}$) of end-points a and b .
- $(a, b)_F := \{x \in \mathbb{F} : a < x \text{ and } x < b\}$, which is the open interval (in $\mathbb{F}$) of end-points a and b .
- $(a, b]_F := \{x \in \mathbb{F} : a < x \text{ and } x \leq b\}$, which is the half-open (open on the right/closed on the left) interval (in $\mathbb{F}$) of end-points a and b .
- $[a, b)_F := \{x \in \mathbb{F} : a \leq x \text{ and } x < b\}$, which is the half-open (open on the right/ closed on the left) interval (in $\mathbb{F}$) of end-points a and b .

We can also introduce the following intervals (in $\mathbb{F}$):

- $(-\infty, b]_F := \{x \in \mathbb{F} : x \leq b\}$;
- $(-\infty, b)_F := \{x \in \mathbb{F} : x < b\}$;
- $[a, +\infty)_F := \{x \in \mathbb{F} : a \leq x\}$;
- $(a, +\infty)_F := \{x \in \mathbb{F} : a < x\}$;
- $(-\infty, +\infty)_F := F$;

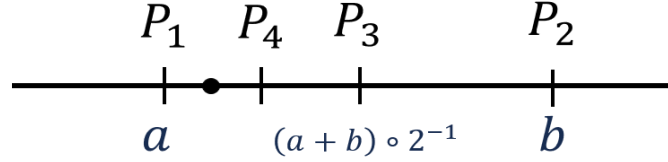
The first four intervals (in $\mathbb{F}$) are called bounded intervals; and the last five intervals (in $\mathbb{F}$) are called unbounded intervals.

Observe that in the definition above we could have $a = b$. Then the $[a, a]_F = \{x \in \mathbb{F} : a \leq x \text{ and } x \leq a\} = \{a\}$, $(a, a)_F = \{x \in \mathbb{F} : a < x \text{ and } x < a\} = \emptyset$. Then in our definition of interval single element and the empty-set are intervals (in $\mathbb{F}$). We call them degenerate intervals (in $\mathbb{F}$); any interval (in $\mathbb{F}$) which is not degenerate we call them, non-degenerate intervals (in $\mathbb{F}$) (only non-degenerate interval contains at least two distinct points).

Problem 2.10. Show that a nondegenerate interval contains infinitely many points.

Solution 2.10. Say $I \subseteq F$ is any non-degenerate interval. For definiteness suppose $I = [a, b]_F$ and $a < b$. Notice that if $2 := 1 + 1 = +(1, 1) \in \mathbb{F}$ and the point $(a + b) \cdot 2^{-1} \in \mathbb{F}$ because $a + b \in \mathbb{F}$

(since $a, b \in \mathbb{F}$ and the sum of any two elements of $\mathbb{F}$ is an element of $\mathbb{F}$) and $2 \neq 0$ hence there exists a multiplicative inverse of 2, and it belongs to $\mathbb{F}$ ($2^{-1} \in \mathbb{F}$). We also have that $a < (a+b) \cdot 2^{-1}$ and $(a+b) \cdot 2^{-1} < b$. Then put $p_1 = a, p_2 = b, p_3 = (a+b) \cdot 2^{-1}, p_4 = (p_1 + p_3) \cdot 2^{-1}, p_5 = (p_4 + p_1) \cdot 2^{-1}, \dots, p_n := (p_1 + p_{n-1}) \cdot 2^{-1}$ for $n > 1$. Each one of these points p_i belong to $\mathbb{F}$ because they are all "middle points" of end-points in I . Then there are infinitely many points in $I = [a, b]_{\mathbb{F}}$



Definition 2.4. Let $\mathbb{F}$ be an ordered field, $a, b \in \mathbb{F}$. Then $\max\{a, b\}$, read the maximum between a, b , is the element of $\mathbb{F}$ defined by

$$\max\{a, b\} := \begin{cases} a & \text{if } b < a; \\ a = b & \text{if } b = a; \\ b & \text{if } a < b. \end{cases}$$

The $\min\{a, b\}$ is the element of $\mathbb{F}$ defined by

$$\min\{a, b\} := \begin{cases} b & \text{if } b < a; \\ a = b & \text{if } b = a; \\ a & \text{if } a < b. \end{cases}$$

Then we define the absolute value of any element x of an ordered field $\mathbb{F}$ being the non-negative $\mathbb{F}$ -number

$$|x| := \max\{x, -x\} = \begin{cases} x, & \text{if } -x < x; \\ -x = x, & \text{if } -x = x; \\ -x, & \text{if } x < -x. \end{cases} = \begin{cases} x, & \text{if } 0 < x; \\ 0, & \text{if } x = 0; \\ -x, & \text{if } x < 0. \end{cases}$$

Definition 2.5. Let $\mathbb{F}$ be any ordered field and $a \in \mathbb{F}, b \in \mathbb{F}$. We say that $|a - b| = |a + (-b)|$ is the distance between the points a and b .

Theorem 2.1. (Triangle Inequality on ordered fields).

For every $a \in \mathbb{F}, b \in \mathbb{F}$, where $\mathbb{F}$ is an ordered field, we have the following

$$|a + b| \leq |a| + |b|.$$

Proof. We want to prove that given $a, b \in \mathbb{F}$ the inequality below holds

$$\max\{a + b, -(a + b)\} \leq \max\{a, -a\} + \max\{b, -b\}$$

Note that either $a > 0$ and $b > 0$, $a > 0$ and $b = 0$, $a > 0$ and $b < 0$, $b > 0$ and $a > 0$, $b > 0$ and $a = 0$, $b > 0$ and $a < 0$, $a = 0$ and $b > 0$, $a = 0 = b$, $a = 0$ and $b < 0$.

With some thinking we observe that we just need to prove the following cases:

- (1) $a > 0, b > 0$;
- (2) $a > 0, b = 0$;
- (3) $a > 0, b < 0$;

$$(4) \ a = 0 = b.$$

For 1. note that $a > 0, b > 0$ we have $|a| = a, |b| = b$. Since $a + b > 0$, $|a + b| = a + b$. Thus $|a + b| = a + b = |a| + |b|$. For 2. we have that $a > 0, b = 0$, then $|a| = a, |0| = 0$ and $a + b = a > 0$.

Therefore $|a + b| = |a + 0| = |a| = a + 0 = |a| + |b|$

For 3. since $a > 0$ and $b < 0$ we have $|a| = a, |b| = -b$. Since $b < 0 < -b$, we have, by adding a to the inequality, that $a + b < a < a - b = |a| + |b|$. But $-a < 0 < a$, hence $-(a + b) < -b < a - b = |a| + |b|$. From the definition

$$|a + b| := \max\{a + b, -(a + b)\} < |a| + |b|$$

□

Now we turn to the study of completeness in an ordered field $\mathbb{F}$. This property will be properly defined shortly, but for now, we examine a classical result attributed to the Pythagorean school. To fully appreciate their contribution, we must first consider how numbers were understood in their time: The ancient Greeks primarily interpreted numbers as lengths of line segments or magnitudes, rather than as abstract algebraic entities as we do today. This geometric perspective strongly influenced their approach to mathematics and underpins many of their insights into the nature of numbers and magnitudes. Before that we need the following lemma

Lemma 2.1. For any integer n , the following statements hold:

- (1) If n is even, then n^2 is even.
- (2) If n is odd, then n^2 is odd.

Furthermore, the converse is also true:

- (1) If n^2 is even, then n is even.
- (2) If n^2 is odd, then n is odd.

Proof. Let's prove these statements:

Case 1: If n is even, then n^2 is even.

If n is even, we can write $n = 2k$ for some integer k . Squaring n , we get:

$$n^2 = (2k)^2 = 4k^2.$$

Since $4k^2$ is clearly divisible by 2, it is even. Thus, if n is even, then n^2 is even.

Case 2: If n is odd, then n^2 is odd.

If n is odd, we can write $n = 2k + 1$ for some integer k . Squaring n , we get:

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

Notice that $2(2k^2 + 2k)$ is even, and adding 1 to it gives an odd result. Therefore, if n is odd, then n^2 is odd.

Now, let's prove the converses:

Case 3: If n^2 is even, then n is even.

Suppose n^2 is even. This means that n^2 is divisible by 2. We proceed by contraposition: if n were odd, we would have $n = 2k + 1$ for some integer k , and squaring it:

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1.$$

This expression is clearly odd because it is of the form $2m + 1$ for some integer m . Hence, if n^2 is even, n must be even.

Case 4: If n^2 is odd, then n is odd.

Suppose n^2 is odd. This means that $n^2 = 2m + 1$ for some integer m . Again, we proceed by contraposition: if n were even, we would have $n = 2k$ for some integer k , and squaring it:

$$n^2 = (2k)^2 = 4k^2.$$

This expression is clearly even, as it is divisible by 2. Hence, if n^2 is odd, n must be odd. $\square$

Theorem 2.2. There exist a non-rational number $\sqrt{2}$.

Proof. Note that in a right triangle of sides $a = b = 1$, and hypotenuse c the hypotenuse can be found by the Pythagorean theorem: $c^2 = 1^2 + 1^2 = x^2 = 2$, where c is the number associated to the length of the hypotenuse of that triangle. We claim that c is not a rational number. For suppose, for a contradiction, that c is rational, so $c = \frac{p}{q}$ for certain $p \in \mathbb{Z}, q \in \mathbb{N}$. Select p, q such that $\gcd(p, q) = 1$, then $\frac{p}{q}$ is said to be an irreducible fraction. Then $c^2 = \frac{p^2}{q^2} = 2$.

Since $p^2 = 2q^2$, p must be even, that is $p = 2k$. So $p^2 = 4k^2 = 2q^2$, and then $q^2 = 2k^2$. Then q is also even, which contradicts $\gcd(p, q) = 1$. Hence there is a non-rational number. $\square$

Problem 2.11. Prove that there is no rational number whose square is 3.

Solution 2.11. Suppose, for the sake of contradiction, that there exist integers p and q such that $\gcd(p, q) = 1$ and

$$\left(\frac{p}{q}\right)^2 = 3.$$

Then, squaring both sides, we obtain:

$$\frac{p^2}{q^2} = 3 \quad \Rightarrow \quad p^2 = 3q^2.$$

This implies that p^2 is divisible by 3. Since 3 is a prime number, if a square is divisible by 3, the number itself must also be divisible by 3. Thus, we conclude that p must be divisible by 3. Therefore, we can write:

$$p = 3k$$

for some integer k . Substituting $p = 3k$ into the equation $p^2 = 3q^2$, we get:

$$(3k)^2 = 3q^2 \quad \Rightarrow \quad 9k^2 = 3q^2 \quad \Rightarrow \quad 3k^2 = q^2.$$

This shows that q^2 is divisible by 3, implying that q must also be divisible by 3.

However, this contradicts our initial assumption that $\gcd(p, q) = 1$, because if both p and q are divisible by 3, their greatest common divisor must be at least 3. Hence, we have reached a contradiction, and we conclude that no such rational number exists whose square is 3.

Problem 2.12. We have learned that every nonempty open interval $I \subset \mathbb{R}$ contains a rational number. Prove that in fact I contains infinitely many rational numbers

Solution 2.12. We have nonempty finite interval $I \subset \mathbb{R}$, (a, b) with finite number of rationals $\{q_1, q_2, q_3, \dots, q_{n-1}, q_n\}$. Then if we choose any pair of rationals from this set, WLOG (q_1, q_2) we will have a rational number. Thus, contradicting our initial statement on infinitely many rationals.

Problem 2.13. Prove that the set $\mathbb{R} \setminus \mathbb{Q}$ of irrationals is dense in $\mathbb{R}$.

Solution 2.13. $\mathbb{Q}$ is dense meaning for all $a, b \in \mathbb{R}$ we have $x \in \mathbb{Q}$ such that $a < x < b$. Given any interval $(a, b) \in \mathbb{Q}$ choose any positive irrational number z , let's say $\sqrt{2}$. By density of rationals we will have rational number x in the interval $(\frac{a}{z}, \frac{b}{z})$ so that xz lies in interval (a, b) and xz is irrational

Problem 2.14. Let A, B , and C be sets.

- (1) Show that $A \sim A$ (relation $\sim$ is reflexive)
- (2) Show that if $A \sim B$, then $B \sim A$ ($\sim$ is symmetric)
- (3) Show that if $A \sim B$ and $B \sim C$, then $A \sim C$ ($\sim$ is transitive)

Solution 2.14. (1) A has same cardinality with itself, so we can denote it by relation $\sim$, $A \sim A$.

- (2) $A \sim B$ means A and B have same cardinality, meaning relation $B \sim A$ also holds.
- (3) $A \sim B, B \sim C$ means $\#A = \#B, \#B = \#C$, thus $\#A = \#C$. Denoted by $A \sim C$.

Problem 2.15. Find the suprema and infima of the following sets.

- (1) $\{\frac{n}{n+1} : n \in \mathbb{N}\}$
- (2) $\{\frac{x}{x+1} : x \in \mathbb{R}, x > 0\}$
- (3) $\{\frac{1}{3n^2+5} : n \in \mathbb{N}\}$
- (4) $\{\frac{n}{m} : m, n \in \mathbb{N}, m+n \leq 10\}$
- (5) $\{2x - x^2 : 0 < x < 2\}$

Solution 2.15. (1) For $n = 1$ we have $\frac{n}{n+1} = \frac{1}{2}$ and it is the smallest value we can get, thus infimum. Observe that $\frac{n}{n+1} < 1$ for all $n \in \mathbb{N}$, thus $\sup(S) = 1, \inf(S) = \frac{1}{2}$

(2) As $x \rightarrow \infty, \frac{x}{x+1} \rightarrow 1$. As $x \rightarrow 0^+, \frac{x}{x+1} \rightarrow 0$. Thus, $\sup(S) = 1, \inf(S) = 0$

(3) If n approaches ∞ , then $3n^2 + 5 \rightarrow \infty$, and consequently $\frac{1}{3n^2+5} \rightarrow 0$. Additionally, we get the greatest value of our set with $n = 1, \frac{1}{8}$. Thus, $\sup(S) = \frac{1}{8}, \inf(S) = 0$

(4) We get the greatest value - $\sup(S) = 9$, with $n = 9, m = 1$, and the lowest value - $\inf(S) = \frac{1}{9}$, with $n = 1, m = 9$.

(5) We can simplify it to $x(2 - x)$. And when $x = 1$, we get $\sup(S) = 1$. If $x \rightarrow 0^+$ or $x \rightarrow 2^-$, we get $\inf(S) = 0$

Problem 2.16. Let $A \subset \mathbb{R}$ be nonempty and bounded above. Let $c \in \mathbb{R}$. Show that $B = \{x + c : x \in A\}$ is bounded above and that $\sup(B) = \sup(A) + c$.

Solution 2.16. $A \subset \mathbb{R}, A \neq \emptyset$. A is bounded above, meaning for all $m \in A$ we have $m < n$. Then for $B = \{x + c : x \in A, c \in \mathbb{R}\}$ we have for all $(m + c) \in B \rightarrow s + c < x + c \rightarrow s < x$, so B is bounded above. Denote $\sup(A) = a$, then for all $\varepsilon > 0$ we have some $m \in A$ such that $a - \varepsilon < m$. Let $\sup(B) = b$, for all $\varepsilon > 0, b - \varepsilon < m + c, b - c - \varepsilon < m$. It means supremum of A is equal to $b - c$, thus $b = a + c$. Done

Problem 2.17. Suppose $A, B \subset \mathbb{R}$ are nonempty and bounded above, with $B \subset A$. Show that $\sup(B) \leq \sup(A)$.

Solution 2.17. $B \subset A$, every element of B is also in A . Let $\sup(A) = a$, for all $\varepsilon > 0, a - \varepsilon < x$, for some $x \in A$. Let $\sup(B) = b$, for all $\varepsilon > 0, b - \varepsilon < x$, for some $x \in B$. Since $B \subset A$ All upper bounds l of B are also upper bounds of A , supremum of A is upper bound of B . Concluding, $\sup(B) \leq \sup(A)$

Problem 2.18. (a) Let $A, B \subset \mathbb{R}$. If $\sup(A) < \sup(B)$, prove that B contains an upper bound for A .

(b) If $\sup(A) \leq \sup(B)$, is it true that B contains an upper bound for A ?

Solution 2.18. (a) $\sup(B)$ is the least upper bound in B , meaning no upper bound in B is less than $\sup(B)$ and it is greater $\sup(A)$, A 's least upper bound. $\sup(A) - \varepsilon < a \leq \sup(A)$. Take $b \in B$ such that $\sup(A) < b \leq \sup(B)$. It gives $a \leq \sup(A) < b$, meaning b is upper bound.

(b) $\sup(A) \leq \sup(B)$

For all $\varepsilon > 0, \sup(A) - \varepsilon < a \leq \sup(A) \leq \sup(B)$

We get $a \leq \sup(B)$, meaning $\sup(B)$ is an upper bound.

Problem 2.19. Let A and B be nonempty subsets of $\mathbb{R}$, bounded above, such that for every $a \in A$ there is $b \in B$ with $a \leq b$. Show that $\sup(A) \leq \sup(B)$

Solution 2.19. In 2.7 we already shown that for $A \subset \mathbb{R}$, and bounded above, $c \in \mathbb{R}$, and $B = \{x + c : x \in A\}$ implies $\sup(B) = \sup(A) + c$. As here $c \geq 0$, we are having $\sup(A) \leq \sup(B)$

Problem 2.20. Let $A \subset \mathbb{R}$ be nonempty and bounded below. Let B be the set of lower bound of A . Show that B is bounded above. What is $\sup(B)$?

Solution 2.20. Let l be the bound of A (means for all $a \in A, a \geq l$) and $B = \{l_1, l_2, \dots, l_n\}$.

Suppose on the contrary there is no upper bound in B . It means for all l_i we will have at least one l_j such that $l_j > l_i$ when l_i is greater than others. But consequently it means l_j is upper bound. Denote $\sup(B) = l_k$, and it means for all $\varepsilon > 0, l_k - \varepsilon < l_i \leq l_k$. $\inf(A) \geq l_i$ for all $l_i \in B$ by definition, and it means $\sup(B) = \inf(A)$

3. SETS(NON-AXIOMATIC), FUNCTIONS

A set is a primitive notion, meaning it is not defined in terms of simpler concepts. In this sense, a set is one of the most fundamental notions in mathematics. The term "collection" is often used synonymously with "set." A set can be viewed as a collection of objects, called its elements. A set may either have elements or may not. If it has elements, it is called a non-empty set. If it has no elements, it is called the empty set, denoted by $\emptyset$.

A set is uniquely determined by its elements, meaning that a set is defined entirely by the objects it contains. This can be done either by explicitly listing the elements of the set or by describing a property that characterizes all of its elements. Given a set, we typically denote it with an uppercase letter from the Latin alphabet, and its elements are denoted by lowercase letters. If A is a set and x is an object, then either x is an element of A , denoted $x \in A$, or x is not an element of A , denoted $x \notin A$.

The **universe of discourse** refers to the set of all elements that are relevant to a particular discussion or problem. It provides the context for the set by specifying the domain or collection of objects under consideration. By fixing a universe of discourse U , we establish the scope of elements being discussed, ensuring that all sets, relations, and properties are properly contextualized.

A set is often defined within this universe by specifying a property $\mathcal{P}(x)$ that characterizes all of its elements. The set then consists of all elements x in the universe U that satisfy the property $\mathcal{P}(x)$. The set can be written as:

$$A = \{x \in U : \mathcal{P}(x)\}$$

This notation indicates that A is the collection of all elements x in the universe U for which the property $\mathcal{P}(x)$ holds true.

Example 3.1. Consider the set of all boys in a particular school. This set contains all male students in that school, and the defining property of the set is that an element of the set is a male student in the school.

Thus, the universe of discourse defines the context in which the set and its elements are understood, ensuring that the set is well-defined within the scope of the problem.

Definition 3.1. (Elementary Notions).

Given two sets A and B , we say that A and B are equal, denoted $A = B$, if and only if every element of A is an element of B and every element of B is an element of A . In other words, for all $x \in A$, we have $x \in B$, and for all $y \in B$, we have $y \in A$.

We say that A is a subset of B , denoted $A \subset B$, if and only if every element of A is also an element of B . We can also say that A is contained in B . When there is at least one element of B that is not an element of A , we say that A is properly contained in B , denoted $A \subsetneq B$.

We say that A is not a subset of B , denoted $A \not\subset B$, if and only if there exists at least one element of A that is not an element of B .

Example 3.2. Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$.

- Every element of A is also in B , so we have $A \subset B$.
- Additionally, $A \neq B$ because B contains the extra element 4. Thus, $A \subsetneq B$ (the inclusion is proper).

Example 3.3. Let $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$.

- $A \cap B = \emptyset$, meaning there are no common elements between A and B .
- Therefore, $A \not\subseteq B$ and $B \not\subseteq A$.

Example 3.4. Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$.

- $A \subseteq B$, meaning that every element of A is in B .
- However, $A \neq B$ since B contains elements 4 and 5 that are not in A , so the inclusion $A \subseteq B$ is proper.

Example 3.5. Let $A = \{a, b, c\}$ and $B = \{a, b, c, d\}$.

- Since there is at least one element in B (namely d) that is not in A , we have $A \subsetneq B$ (the inclusion is proper).

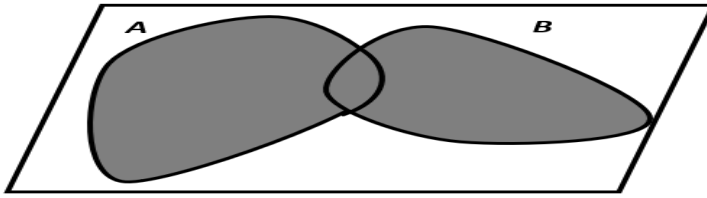
Now we define some "operations" one can perform with sets

Definition 3.2. (Set theoretical Operations).

Given sets A, B we define

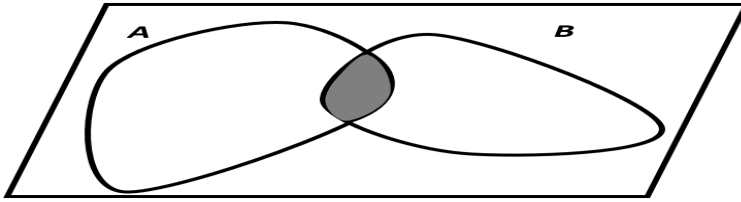
- (1) Their union (logical sum) being the set of all elements belonging an either A or B , that is

$$A \cup B := \{x : x \in A \text{ or } x \in B\}.$$



- (2) Their intersection (logical product) being the set of all elements that belong to both A and B , that is

$$A \cap B := \{x : x \in A \text{ and } x \in B\}.$$



If $A \cap B = \emptyset$ we say that A and B are disjoint.

- (3) The difference (logical negation) of B and A is the set of all elements of B which do not belong to A , that is

$$B - A := B \setminus A := \{x : x \in B \text{ and } x \notin A\}.$$

When $A \subseteq B$ we say that $B - A$ is the (logical) complement of A relative to B and we denote it by $C_B A$ (or A^c) when it is clear that the complement is taken with respect to B .

- (4) A cartesian B to be the set of ordered pairs (a, b) such that the first entry of (a, b) , that is a , is an element of A . In symbols it is written as

$$A \times B := \{(a, b) : a \in A \text{ and } b \in B\}$$

We say that $(a, b) = (c, d)$ if, and only if, $a = c$ and $b = d$.

Problem 3.1. Prove each one of the following operating rules for sets:

- $A \cap B = B \cap A; A \cup B = B \cup A$

Proof. Let $x \in A \cup B$, then $x \in A$ or $x \in B$. Definitely $x \in B$ or $x \in A$, if it wasn't true then $x \notin B$ and $x \notin A$, but we know that $x \in A$ or $x \in B$, which makes $x \notin B$ and $x \notin A$ false. Hence $x \in B$ or $x \in A$, meaning $x \in B \cup A$. The other inclusion, that is $B \cup A \subseteq A \cup B$ can be proven using the same ideas. So $A \cup B = B \cup A$.

Now if $y \in A \cap B$ we have that $y \in A$ and $y \in B$. We claim that $y \in B$ and $y \in A$. For if it wasn't the case then $y \notin B$ or $y \notin A$. But we know that y belongs to both A and B . Therefore the assumption that $y \notin B$ or $y \notin A$ must be false, in other words $y \in B$ and $y \in A$. This proves that $A \cap B \subseteq B \cap A$.

- $A \cap (B \cap C) = (A \cap B) \cap C$

Proof. For $A \cap (B \cap C)$ we have $x \in A$ and $x \in (B \cap C)$, and it means $x \in A$ and ($x \in B$ and $x \in C$). On the other hand, for $(A \cap B) \cap C$ we have $x \in (A \cap B)$ and $x \in C$, and $x \in A$ and $x \in B$ and $x \in C$. If it is not true we will have $x \notin A$ or $x \notin B$ or $x \notin C$, which fully contradicts the first statement. So, equality holds.

- $A \cup (B \cup C) = (A \cup B) \cup C$

Proof. For the left-hand-side we have $x \in A$ or $x \in (B \cup C)$, which means $x \in A$ or $x \in B$ or $x \in C$. For the right-hand side we have $x \in (A \cup B)$ or $x \in C$. If the first statement is false, then we have $x \notin A$ and $x \notin B$ and $x \notin C$, which definitely contradicts the right-hand-side statement, proving our equality.

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof. Meaning of the left-hand-side is, $x \in A$ and $x \in B$ or $x \in C$. We want to prove that it is equal to $(x \in A \text{ and } x \in B)$ or $(x \in A \text{ and } x \in C)$. Assume by contradiction that $(x \notin A \text{ or } x \notin B)$ and $(x \notin A \text{ or } x \notin C)$. But we know that x is at least in A . So our assumption is false, and equality holds.

- $C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B)$

Proof. *LHS* means $x \in C$ and $(x \notin A \text{ and } x \notin B)$. Assuming the inverse of the right-hand-side is true, we get that $(x \notin C \text{ or } x \in A)$ and $(x \notin C \text{ or } x \in B)$. But from *LHS* we already know that $x \in C$, so our assumption is false and equality is proved.

Problem 3.2. Find what are the following sets

- (1) $A \times (B \cup C)$

$$B \cup C := \{x : x \in B \text{ or } x \in C\}$$

$$A \times (B \cup C) = \{(a, b) : a \in A \text{ and } (b \in B \text{ or } b \in C)\}$$

- (2) $A \times (B \cap C)$

$$B \cap C := \{x : x \in B \text{ and } x \in C\}$$

$$A \times (B \cap C) = \{(a, b) : a \in A \text{ and } (b \in B \text{ and } b \in C)\}$$

- (3) $A \times (C - B)$

$$C - B := \{x : x \in C \text{ and } x \notin B\}$$

$$A \times (C - B) = \{(a, b) : a \in A \text{ and } (b \in C \text{ and } b \notin B)\}$$

- (4) $(C - A) \times (B - C)$

$$C - A := \{x : x \in C \text{ and } x \notin A\}$$

$$B - C := \{x : x \in B \text{ and } x \notin C\}$$

$$(C - A) \times (B - C) = \{(a, b) : (a \in C \text{ and } a \notin A) \text{ and } (b \in B \text{ and } b \notin C)\}$$

Definition 3.3. A function between sets A and B is a triple (A, f, B) , consisting of a domain (source set A), a codomain (target set B), and a rule f^1 , which associates every element of A with exactly one element of B . For practicality, we denote a function (A, f, B) by $f : A \rightarrow B$.

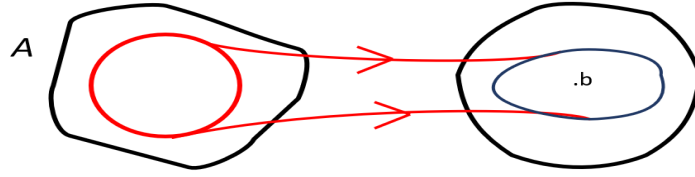
Two functions (A, f, B) and (C, g, D) are equal if and only if $A = C$, $B = D$ (equality of sets), and f and g assign the same output for every input.

Given a function $f : A \rightarrow B$, we say that $y \in B$ is the image of an element $x \in A$ if and only if $y = f(x)$. The image set of the function f is the set of all $y \in B$ such that $y = f(x)$ for some $x \in A$. In symbols:

$$f(A) := \{y \in B : \text{for some } x \in A, y = f(x)\} = \{f(x) : x \in A\}.$$

We also say that $f(A)$ is the image of A under f . If $C \subseteq A$, the image of C under f is defined as:

$$f(C) := \{f(x) : x \in C\}.$$



Let $D \subseteq B$ and $f : A \rightarrow B$ be a function. The preimage of D under f consists of all $x \in A$ such that $f(x) \in D$. In symbols:

$$f^{-1}(D) := \{x \in A : f(x) \in D\}.$$

The preimage of an element $b \in B$, denoted $f^{-1}(\{b\})$, consists of all $x \in A$ such that $f(x) = b$. In symbols:

$$f^{-1}(\{b\}) := \{x \in A : f(x) = b\}.$$

Example 3.6. Let $A = \{1, 2\}$, $B = \{7, 8, 11, 27\}$. Consider the function $f : A \rightarrow B$ given by $f(1) := 27$ and $f(2) := 8$. What is the image set of f ? What is $f^{-1}(\{7\})$, $f^{-1}(\{7, 8, 27\})$?

The image set of f is the set of all image points of A . These are the values $f(x)$ for $x \in A$. We have $f(1) = 27$ and $f(2) = 8$, so the image set is:

$$f(A) = \{8, 27\}.$$

The preimage of a set $S \subseteq B$ consists of all $x \in A$ such that $f(x) \in S$. For $f^{-1}(\{7\})$, since there is no $x \in A$ with $f(x) = 7$, we have:

$$f^{-1}(\{7\}) = \emptyset.$$

For $f^{-1}(\{7, 8, 27\})$, we find all $x \in A$ such that $f(x) \in \{7, 8, 27\}$. We know that $f(1) = 27$ and $f(2) = 8$, so:

$$f^{-1}(\{7, 8, 27\}) = \{1, 2\}.$$

Example 3.7. Consider $f(x) := \frac{x^2-4}{x+2}$ for every $x \neq -2$, and here the codomain is $\mathbb{R}$. Find the image set of f . The inverse image of the set $I = [2, 200)$, and of $J = \{2n : n \in \mathbb{Z}\}$

Note that $f^{-1}(I) = [4, 202)$, because for all $x \in [4, 202)$ we have $f(x) = x - 2 \in [2, 200)$. For $f^{-1}(\{2n : n \in \mathbb{Z}\})$ notice that $x \in f^{-1}(\{2n : n \in \mathbb{Z}\})$ means $f(x) \in \{2n : n \in \mathbb{Z}\}$, that is $f(x) = 2n$ for certain $n \in \mathbb{Z}$. But $f(x) = x - 2$ then $x - 2 = 2n$, so $x = 2 + 2n$ and $x \neq -2$. Hence $f^{-1}(\{2n : n \in \mathbb{Z}\}) = \{2 + 2n : n \in \mathbb{Z} \mid \{-4\}\}$.

¹Note that the rule f does not need to be expressed as a formula; it can simply be given as a descriptive statement.

Definition 3.4. Regarding the nature of the rule/law of definition of $f : A \longrightarrow B$, we have 3 important types:

- One-to-one functions / injective functions:

We say that f is a one-to-one function or that it is injective if and only if the following statement is true:

If $f(a) = f(\tilde{a})$ for certain $a, \tilde{a} \in A$, then $a = \tilde{a}$, or that for all $a, \tilde{a} \in A$ such that $a \neq \tilde{a}$, we have $f(a) \neq f(\tilde{a})$.

- Onto functions / surjective functions:

We say that f is onto or that it is surjective if and only if for every element $b \in B$, there exists an element $a \in A$ such that $f(a) = b$. In other words, the image of f is the entire set B , or $f(A) = B$.

- One-to-one correspondences / bijective functions:

We say that f is a bijective function if and only if it is both injective and surjective. That is, for every $b \in B$, there exists exactly one $a \in A$ such that $f(a) = b$. In other words, f is a one-to-one correspondence between the sets A and B .

Example 3.8. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) := x^3$ for every $x \in \mathbb{R}$. Note that f is an injective function because if we have $f(x_1) = f(x_2)$ for certain $x_1, x_2 \in \mathbb{R}$, we have $x_1^3 = x_2^3$, or equivalently $x_1^3 - x_2^3 = 0$. This can be factored as:

$$x_1^3 - x_2^3 = (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2).$$

Now, if $x_1 \neq x_2$, then the factor $x_1^2 + x_1x_2 + x_2^2$ is strictly positive. To see this, observe that:

$$x_1^2 + x_1x_2 + x_2^2 = \frac{1}{2} \left((x_1 - x_2)^2 + (x_1 + x_2)^2 \right).$$

Both terms on the right-hand side are non-negative, and the first term is strictly positive unless $x_1 = x_2$. Hence, $x_1^2 + x_1x_2 + x_2^2 > 0$ whenever $x_1 \neq x_2$.

Therefore, if $f(x_1) = f(x_2)$, we must have $x_1 = x_2$, which proves that f is injective.

Example 3.9. (The Matching Socks).

Imagine you have a basket of socks with 5 different colors: red, blue, green, yellow, and purple. You are tasked with pairing them up with 5 different shoes, each corresponding to a specific color.

- The Set of Socks: {red, blue, green, yellow, purple}
- The Set of Shoes: {red shoe, blue shoe, green shoe, yellow shoe, purple shoe}

The Bijection (One-to-One Correspondence): You can assign each sock to exactly one shoe and vice versa, without any repetition or omissions. For example:

- Red sock goes with red shoe
- Blue sock goes with blue shoe
- Green sock goes with green shoe
- Yellow sock goes with yellow shoe
- Purple sock goes with purple shoe

In this case:

- Injectivity: No two socks share the same shoe. If two socks were to share the same shoe, it would violate the one-to-one nature of the function.
- Surjectivity: Every shoe has exactly one sock. No shoe is left without a matching sock, so the function is onto.

Since the function is both injective (one-to-one) and surjective (onto), we have a bijection. There's a perfect pairing between socks and shoes.

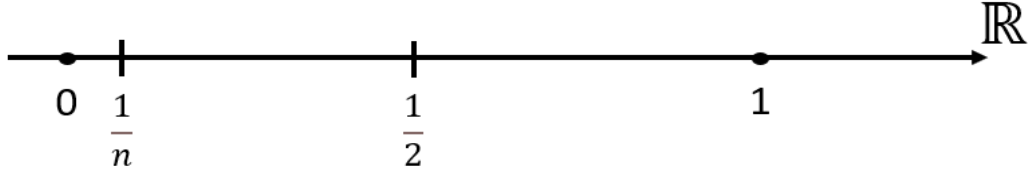
Problem 3.3. Prove that if $f : A \longrightarrow B$ is a one-to-one (injective) function then the function $\tilde{f} : A \longrightarrow f(A)$ given by $\tilde{f}(x) := f(x)$ for all $x \in A$ is a one-to-one correspondence (bijective).

Solution 3.1. For suppose that $\tilde{f}(x) = \tilde{f}(y)$ for certain $x, y \in A$. We have, by the definition of $\tilde{f}$, that $f(x) = f(y)$ and using that f is a one-to-one function (injective) we conclude that $x = y$. Then $\tilde{f}$ must be a one-to-one function. In order to prove surjectivity of $\tilde{f}$, we must show that $\tilde{f}(A) = f(A)$. By definition of function the following inclusion is immediate $\tilde{f}(A) \subset f(A)$, the other inclusion is the one we need to work a little more. If $y \in f(A)$ there exists $x \in A$ such that $y = f(x)$, thus $\tilde{f}$ is surjective.

4. COMPLETE ORDERED FIELDS

Definition 4.1. Let $\mathbb{F}$ be any ordered field, and $H \subseteq \mathbb{F}$. We say that H is bounded (in $\mathbb{F}$) if, and only if, there exists $a, b \in \mathbb{F}$ for which $H \subseteq [a, b]_{\mathbb{F}}$. If H is not bounded we call it unbounded (in $\mathbb{F}$).

Example 4.1. Consider $\mathbb{F} = \mathbb{R}$ and $H := \{\frac{1}{n} : n \in \mathbb{N}\}$. Note that for all $n \in \mathbb{N}$, $\frac{1}{n} \leq 1$ and $\frac{1}{n} > 0$. Therefore every $x \in H$ is in $(0, 1]$, meaning $H \subseteq (0, 1]$ so H is bounded (in $\mathbb{R}$)



Definition 4.2. Let $\mathbb{F}$ be any ordered field, and $H \subseteq \mathbb{F}$. We say that H is bounded above (in $\mathbb{F}$) if, and only if, there exists some $l \in \mathbb{F}$ such that for all $x \in H$ we have $x \leq l$. Any $\mathbb{F}$ -number $v \in \mathbb{F}$ such that for all $x \in H$ the $x \leq v$ is called an upper bound of H (in $\mathbb{F}$). In practical terms H is bounded above if, and only if, there is $l \in \mathbb{F}$ such that $H \subseteq (-\infty, l]$.

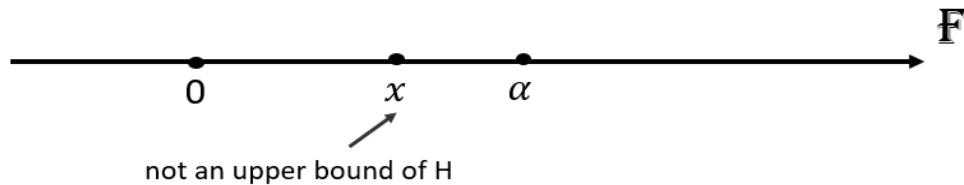
Similarly we say that H is bounded below (in $\mathbb{F}$) if, and only if, there exists some $m \in \mathbb{F}$ such that for all $x \in H$ the $m \leq x$. Any $\mathbb{F}$ -number $\theta \in \mathbb{F}$ such that for all $x \in H$ we have $\theta \leq x$ is called a lower bound of H (in $\mathbb{F}$).

Definition 4.3. Let $\mathbb{F}$ be any ordered field, and $H \subseteq \mathbb{F}$. We say that a number $\alpha \in \mathbb{F}$ is a least upper bound of H if, and only if, α is an upper bound of H and for every upper bound v of H we have $\alpha \leq v$.

We can also call α the supremum of H (in $\mathbb{F}$) and denote it by $\alpha = \text{lub}(H) = \sup(H)$

It must be observed that α is unique for if $\tilde{\alpha}$ was a least upper bound of H , $\tilde{\alpha}$ is an upper bound of H , and if v is an upper bound of H $\tilde{\alpha} \in \mathbb{F} \leq v$. In particular α is an upper bound of H , hence $\tilde{\alpha} \leq \alpha$. Similarly we have $\alpha \leq \tilde{\alpha}$. So $\alpha = \tilde{\alpha}$.

Note that if $H \subseteq \mathbb{F}$ and α is its least upper bound then $x < \alpha$ cannot be an upper bound of H . But x not being an upper bound for H means that not all elements of H are less than or equal to x , so there is (at least one) element of H , say h , such that $x < h \leq \alpha$. Therefore α is a least upper bound of H if, and only if, for every $x < \alpha$ there exists $h \in H$ such that $x < h \leq \alpha$.



Proposition 4.1. In any ordered field $\mathbb{F}$, where $H \subseteq \mathbb{F}$, $\alpha \in \mathbb{F}$ is the least upper bound of H if, and only if, for every $\varepsilon > 0$ there exists some $h \in H$ such that $\alpha - \varepsilon < h \leq \alpha$.

Proof. ($\Longleftrightarrow$) Say that $\alpha \in \mathbb{F}$ is a least upper bound of H . Then, by definition no $x < \alpha$ is an upper bound of H , that is, there exists (at least one) element $\delta \in \mathbb{F}$ such that $x < \delta \leq \alpha$. Then given any $\varepsilon > 0$ we know that $\alpha - \varepsilon < \alpha$ therefore $\alpha - \varepsilon$ is not an upper bound of H , so there exists $h \in H$ such that $\alpha - \varepsilon < h \leq \alpha$. Thus the first implication is proven.

($\Longleftarrow$) Suppose that for all $\varepsilon > 0$ there exists $h \in H$ so that $\alpha - \varepsilon < h \leq \alpha$. But then if $x < \alpha$, note that for $\varepsilon_x = \alpha - x > 0$ we have that $x = \alpha - \varepsilon_x < \alpha$. Then for $\varepsilon = \varepsilon_x > 0$ there exists $h_x \in H$ such that $x = \alpha - \varepsilon_x < h_x \leq \alpha$. So α must be a least upper bound. $\square$

Definition 4.4 ((infimum / greatest lower bound)). Let $\mathbb{F}$ be any ordered field, and $H \subseteq \mathbb{F}$. We say that a number $\beta \in \mathbb{F}$ is a greatest lower bound of H if, and only if, β is a lower bound of H and if η is a lower bound of H we have that $\eta \leq \beta$. Thus no $x < \beta$ can be a lower bound of H . We can also call β the infimum of H (in $\mathbb{F}$) and we denote $\beta := \inf(H) = \text{glb}(H)$.

Similarly to the supremum / least upper bound we can prove that if the infimum / greatest lower bound of H exists then it is uniquely determined.

Proposition 4.2. In any ordered field $\mathbb{F}$, where $H \subseteq \mathbb{F}$, $\beta \in \mathbb{F}$ is the greatest lower bound of H if, and only if, for every $\varepsilon > 0$ there exists some $h \in H$ such that $\beta \leq h < \beta + \varepsilon$.

Proof. ($\implies$) Suppose that $\beta \in \mathbb{F}$ is the greatest lower bound of H , then for all $h \in H$ we have $\beta \leq h$ (β is a lower bound of H) and for every lower bound v of H we have $v \leq \beta$.

Given any $\varepsilon > 0$ the $\beta + \varepsilon \in \mathbb{F}$ cannot be a lower bound of H , because all lower bound of H are less than or equal to β and $\beta < \beta + \varepsilon$. Therefore there must be (at least one) $h \in \mathbb{F}$ for which $h < \beta + \varepsilon$. And since β is a lower bound of H , $\beta \leq h$. First implication is proven.

($\impliedby$) Suppose for all $\varepsilon > 0$ there exists $h \in H$ such that $\beta \leq h < \beta + \varepsilon$. But then if $x > \beta$ note that for $\varepsilon_x = x - \beta$ we have that $x = \varepsilon_x + \beta \geq \beta$. Thus, for $\varepsilon = \varepsilon_x > 0$ there exists $h_x \in H$ such that $x = \varepsilon_x + \beta > h_x \geq \beta$. So β must be a greatest lower bound. $\square$

There exists an ordered field $\mathbb{R} \supseteq \mathbb{Q}$ such that for every non-empty subset of $\mathbb{R}$, say B , bounded above then B has a least upper bound (supremum) and it belongs to $\mathbb{R}$.

We call $\mathbb{R}$ the set of real numbers (when we view it as a set) or real field (when we view it as an ordered field).

Proposition 4.3. In $\mathbb{R}$ every non-empty subset $\mathbb{B}$ of $\mathbb{R}$, which is bounded below (in $\mathbb{R}$) has a greatest lower bound and it is a real number.

Proof. Let $\mathbb{B} \subseteq \mathbb{R}$ be non-empty and bounded below. Define $-\mathbb{B} := \{-b \in \mathbb{R} : b \in \mathbb{B}\}$. This $-\mathbb{B} \neq \emptyset$ because $B \neq \emptyset$ and since $\mathbb{B}$ is bounded below there exists $x \in \mathbb{R}$ for which statement $x \leq b$ is true for all $b \in \mathbb{B}$. But this is equivalent to $-b \leq -x$ for all $b \in \mathbb{B}$. Therefore $-\mathbb{B}$ is bounded above. By the completeness axiom we know that every non-empty subset of $\mathbb{R}$ which is bounded above has a least upper bound (supremum) in $\mathbb{R}$. Hence the $\lambda := \text{lub}(-\mathbb{B}) = \sup(-\mathbb{B})$ exists in $\mathbb{R}$ and it means that for all $-b \in -\mathbb{B}$, for $b \in \mathbb{B}$ we have $-b \leq \lambda$ for all $b \in \mathbb{B}$ and if $\nu \in \mathbb{R}$ is an upper bound of $-\mathbb{B}$ we have $\lambda \leq \nu$. As a result $-\lambda \leq b$ for all $b \in \mathbb{B}$ meaning that $-\lambda$ is a lower bound for $\mathbb{B}$. If $\theta \in \mathbb{R}$ is a lower bound of $\mathbb{B}$ notice that $-\theta \in \mathbb{R}$ is an upper bound of $-\mathbb{B}$ because $\theta \leq b$ for all $b \in \mathbb{B}$, so $-b \leq -\theta$ for all $b \in \mathbb{B}$. We know for a fact that any upper bound of $\mathbb{B}$ is greater than or equal to λ , thus $\lambda \leq -\theta$, or equivalently $\theta \leq -\lambda$. This shows that $-\lambda$ is the greatest lower bound (supremum) of $\mathbb{B}$ and that

$$-\sup(-\mathbb{B}) = -\text{lub}(-\mathbb{B}) = -\lambda = \text{glb}(\mathbb{B}) = \inf(B)$$

$\square$

The proposition above guarantees that the completeness axiom in $\mathbb{R}$ also guarantees the existence of glb/inf for non-empty subsets of $\mathbb{R}$ that are bound.

Why the need for completeness? At first we need it for practical reasons. For example we consider the subset of positive rationals $\{r \in \mathbb{Q}^+ : r^2 > 2\} = \mathbb{K}$. It is clear that $2 \in \mathbb{K}$ because $2 \in \mathbb{Q}^+$ and $2^2 = 4 > 2$, so $\mathbb{K} \neq \emptyset$. Is it bounded below in $\mathbb{Q}$? Notice that $1 \notin \mathbb{K}$ and every $r \in \mathbb{K}$ satisfies the following $1 < r$, otherwise if there was $r_0 \in \mathbb{K}$ such that $r_0 < 1$ then we would have $r_0^2 < 1$, but since $r_0 \in \mathbb{K}$ then $r_0^2 > 2$. This contradiction guarantees that $\mathbb{K}$ is bounded below (in $\mathbb{Q}$).

Hence it is non-empty and bounded below. But it does not have greatest lower bound in $\mathbb{Q}$. For suppose that $\mathbb{K} = \{r \in \mathbb{Q}^+ : r^2 > 2\}$ has a rational greatest lower bound, let's call it α . Either $\alpha^2 > 2$ or $\alpha^2 < 2$, $\alpha^2 = 2$ is impossible.

If $\alpha^2 > 2$ consider $\alpha + \frac{1}{n}$, where we want to determine the $n \in \mathbb{N}$ such that all $r \in \mathbb{K}$ satisfy $\alpha + \frac{1}{n} \leq r$. For this to be true the following must happen, $2 < (\alpha + \frac{1}{n})^2 \leq r^2$.

Notice that $(\alpha + \frac{1}{n})^2 = \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} > \alpha^2 + \frac{2\alpha}{n}$. If we make $\alpha^2 + \frac{2\alpha}{n} > 2$, then $(\alpha + \frac{1}{n})^2 > 2$. For $\alpha^2 + \frac{2\alpha}{n} > 2$ we must pick $n \in \mathbb{N}$ such that $n > \frac{2\alpha}{\alpha^2 - 2}$ (in fact solving the inequality $\alpha^2 + \frac{2\alpha}{n} > 2$ for n we find that n must be taken as above).

Then we found that $\alpha + \frac{1}{n} \in \mathbb{Q}$ is also a lower bound of $\mathbb{K}$, contradicting the fact that α is infimum/ glb of $\mathbb{K}$.

Theorem 4.1 (Archimedean property). The following statements are (logically) equivalent

- (1) $\mathbb{N}$ are not bounded above in $\mathbb{R}$.
- (2) For every $x > 0$ there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < x$.
- (3) For every $x \in \mathbb{R}, y \in \mathbb{R}$ such that $y > 0$ there exists $n \in \mathbb{N}$ such that $x < ny$.

Proof. We have to prove that (i) $\iff$ (ii) and (ii) $\iff$ (iii). For that we prove that (i) $\implies$ (ii), then that (ii) $\implies$ (iii), finally that (iii) $\implies$ (i).

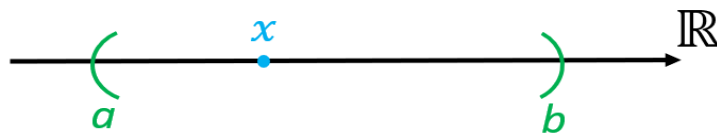
Assume (i). That is, $\mathbb{N}$ is not bounded above in $\mathbb{R}$. Note that if $\mathbb{N}$ was bounded above in $\mathbb{R}$ there would be some $\zeta \in \mathbb{R}$ such that for all $n \in \mathbb{N}$ we would have $n \leq \zeta$. Then $\mathbb{N}$ not being bounded above in $\mathbb{R}$ means that for every $\lambda > 0$ there exists $n_0 \in \mathbb{N}$ such that $\lambda < n_0$. Now given any $x > 0$, we have that $\frac{1}{x} > 0$. Therefore, by the previous observation there must be $n \in \mathbb{N}$ such that $\frac{1}{x} < n$, in other words there exists $n \in \mathbb{N}$ such that $\frac{1}{x} < n$, and this shows (i) $\implies$ (ii).

Now suppose that (iii) holds, that is, for every $a > 0$ there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < a$. Given any $x \in \mathbb{R}, y \in \mathbb{R}$ such that $y > 0$ we want $n \in \mathbb{N}$ such that $x < n_0 y$. By *check* either $x < 0, x = 0$ or $x > 0$. If $x \leq 0$ note that $ny > 0$ for all $n \in \mathbb{N}$ because $y > 0$ and of mathematical induction then there is nothing to prove. So we can suppose without loss of generality that $x > 0$. Note that by the properties of an ordered field the $\frac{y}{x} > 0$. Then applying (ii) for $\frac{y}{x} = y \cdot x^{-1}$ we find $n_0 \in \mathbb{N}$ such that $0 < \frac{1}{n_0} < \frac{y}{x}$. That is, we found $n_0 \in \mathbb{N}$ with $x < n_0 y$. So (ii) $\implies$ (iii).

Now assume (iii). Then for any $\zeta > 0$ there exists $n \in \mathbb{N}$ such that $n = n \cdot 1 > \zeta$. If $\mathbb{N}$ was bounded above in $\mathbb{R}$ this couldn't happen, then the only conclusion we can make is that $\mathbb{N}$ is not bounded above (in $\mathbb{R}$).

Finally we have (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (i), and we are done. $\square$

Definition 4.5. Let $X \subseteq \mathbb{R}$. We say that X is dense in $\mathbb{R}$ if, and only if, for every non-empty open interval I if $\mathbb{R}$ we have $X \cap I \neq \emptyset$. That is for every $a, b \in \mathbb{R}$ with $a < b$ we have $X \cap (a, b) \neq \emptyset$; or that for every $a, b \in \mathbb{R}$ there exists some $x \in X$ such that $a < x < b$.



Theorem 4.2. $\mathbb{Q}$ is dense in $\mathbb{R}$, that is, any two real numbers x, y with $x < y$ there exists (at least) one $r \in \mathbb{Q}$ such that $x < r < y$.

Proof. Note that $x > 0, y > 0$ or $x = 0, y > 0$ or $x < 0, y > 0$, or $x > 0, y = 0$, or $x > 0, y < 0$, or $x = 0, y < 0$, or $x < 0, y < 0$, or $x < 0, y = 0$. We don't need to cover all cases because they are all the same. Let's suppose that $x, y > 0$.

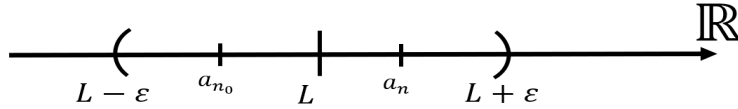
So we have $0 < x < y$. Note that $y - x > 0$ then by the Theorem above (Archimedean property) there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < y - x$ (length of $[x, y]$). Consider the set of all $\{k \in \mathbb{N} \mid k > nx\}$. Note that this subset of $\mathbb{N}$ is non-empty because $\mathbb{N}$ is not bounded in $\mathbb{R}$. By the well-ordering principle there exists smallest $m \in \mathbb{N}$ such that $m > nx$. Then $\frac{m}{n} > x$. $m - 1$ is not bigger than nx because m is the smallest natural bigger than nx . So $m - 1 < nx$, and since $1 < ny - nx$ we have $m < 1 + nx < ny$, this shows us that $m < ny$, or that $\frac{m}{n} < y$. Then we have a rational $\frac{m}{n}$ such that $x < \frac{m}{n} < y$. We are done. $\square$

5. NUMERICAL SEQUENCES (IN $\mathbb{R}$)

Definition 5.1. A sequence of real numbers is any function $a: \mathbb{N} \rightarrow \mathbb{R}$. We denote the sequence $a: \mathbb{N} \rightarrow \mathbb{R}$ by $(a_n)_{n \in \mathbb{N}}$. The $a_n := a(n)$ is called the n -th term of sequence.

We are studying sequences because they are the most elementary objects to introduce the idea of limit of a variable quantity. This topic is fundamental because the theory of limits will be developed using sequences.

Definition 5.2. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of real number, and $L \in \mathbb{R}$. We say that L is a limit of the sequence (a_n) when n tends to $+\infty$ if, and only if, for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that $|a_n - L| < \varepsilon$ for all $n \in \mathbb{N}$ such that $n \geq n_0$.



Note that if $(a_n)_{n \in \mathbb{N}}$ has a limit L it must be unique. In fact if M was also a limit of (a_n) , for each $\varepsilon > 0$ there are $n_0 \in \mathbb{N}$, $n'_0 \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$ for all $n \geq n_0$ and $|a_n - M| < \varepsilon$ for all $n \geq n'_0$. Then if $k_0 = \max\{n_0, n'_0\}$ whenever $n \geq k_0$ we have

$$|L - M| = |(a_n - M) - (a_n - L)| \leq |a_n - M| + |a_n - L| < \varepsilon + \varepsilon = 2\varepsilon$$

So we have proven that for every $\varepsilon > 0$ we have $|L - M| < 2\varepsilon$. Then question arises: is it possible for $0 < |L - M|$?

Suppose $|L - M| > 0$, then choosing $\varepsilon' = \frac{|L-M|}{2} > 0$ we have $|L - M| < 2\varepsilon' = |L - M|$. Hence $|L - M| = 0$, which shows that $L = M$ and that there can be only one limit of (a_n) . Then we denote $L = \lim_{n \rightarrow \infty} a_n$. When the limit of a sequence exists we say that it converges to that limit or that the sequence is convergent.

Example 5.1. Let $L \in \mathbb{R}$ and $(a_n)_{n \in \mathbb{N}}$ given by $a_n := L$ for all $n \in \mathbb{N}$. Note that this sequence, called constant sequence equal to L , is convergent.

Given any $\varepsilon > 0$, since we have $|a_n - L| = 0$ for all $n \in \mathbb{N}$ we can choose any $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, then $|a_n - L| < \varepsilon$.

Example 5.2. Define $(a_n)_{n \in \mathbb{N}}$ by $a_n := \frac{1}{n}$ for all $n \in \mathbb{N}$. Does $(a_n)_{n \in \mathbb{N}}$ converge? If so find its limit, then prove that this is the limit of $(\frac{1}{n})_{n \in \mathbb{N}}$.

Fix any $\varepsilon > 0$ we want a $n_0 \in \mathbb{N}$ such that $|a_n - 0| = \frac{1}{n} < \varepsilon$ whenever $n \geq n_0$. Since $\mathbb{R}$ has the Archimedean property there exists $n_0 \in \mathbb{N}$ such that $1 < n_0 \cdot \varepsilon$ and if $n \geq n_0$ we have that $1 < n_0 \varepsilon \leq n \varepsilon$, that is $\frac{1}{n} \leq \frac{1}{n_0} < \varepsilon$ for all $n \geq n_0$. This shows that $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$.

Example 5.3. Define $(a_n)_{n \in \mathbb{N}}$ by $a_n := \sqrt[n]{n}$ for all $n \in \mathbb{N}$. Does $(a_n)_{n \in \mathbb{N}}$ converge? If so find its limit, then prove that this is the limit of $(\sqrt[n]{n})_{n \in \mathbb{N}}$.

For this limit we consider this trick, put $a_n - 1 = r_n$. Note that $a_n > 0$ for all $n \in \mathbb{N}$, then $r_n = a_n - 1 > -1$ for all $n \in \mathbb{N}$. Proving by induction on n that $r_n \geq 0$ for all $n \in \mathbb{N}$ we have by Newton's binomial theorem that

$$a_n^n = (1 + r_n)^n = 1 + nr_n + \frac{n(n-1)r_n^2}{2!} + \dots + r_n^n \geq 1 + \frac{n(n-1)r_n^2}{2}$$

Then $n \geq 1 + \frac{n(n-1)}{2} r_n^2$ for all $n \in \mathbb{N}$ and it tell us that $\sqrt{\frac{2}{n}} \geq r_n$ for all $n \in \mathbb{N}$. But $\lim_{n \rightarrow \infty} \sqrt{\frac{2}{n}} = 0$ therefore for each $\varepsilon > 0$ one fixes there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $|\sqrt{\frac{2}{n}} - 0| = \sqrt{\frac{2}{n}} < \varepsilon$ for all $n \geq n_0$. Since $\sqrt{\frac{2}{n}} \geq r_n$ we have for all $n \in \mathbb{N}$ such that $n \geq n_0$ the following inequality $r_n < \varepsilon$. And

using the definition of $r_n := a_n - 1$ and that $r_n \geq 0$ we have just proven that $|a_n - 1| = |r_n| = r_n < \varepsilon$ whenever $n \geq n_0$. This shows that for every $\varepsilon > 0$ one chooses there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $|a_n - 1| < \varepsilon$ for all $n \in \mathbb{N}$ such that $n \geq n_0$, in other words $\lim_{n \rightarrow \infty} a_n = 1$

Example 5.4. The sequence $(a_n)_{n \in \mathbb{N}}$ given by $a_n = \frac{1}{2^n}$ for all $n \in \mathbb{N}$ converges and its limit is zero.

Note that for all $n \in \mathbb{N}$ we have $2^n \geq n$. By induction on n , we see that statement hold for $n = 1$ and if it is true for certain $n_0 \in \mathbb{N}$, then $2^{n_0+1} = 2 \cdot 2^{n_0} \geq 2n_0 \geq n_0 + 1$. Therefore we have proven that $2^n \geq n$ for all $n \in \mathbb{N}$. As a result $\frac{1}{n} \geq \frac{1}{2^n}$ for all $n \in \mathbb{N}$. And since $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ for each $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$ whenever $n \geq n_0$. We conclude that $\frac{1}{2^n} < \varepsilon$ for all $n \geq n_0$. This proves that $\lim_{n \rightarrow \infty} \frac{1}{2^n} = 0$

Definition 5.3. A sequence of real numbers $(a_n)_{n \in \mathbb{N}}$ is bounded (in $\mathbb{R}$) if, and only if, the set $\{a_n \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$ is bounded (in $\mathbb{R}$). In other words there exists $x, y \in \mathbb{R}$ with $x \leq y$ such that $x \leq a_n \leq y$ for all $n \in \mathbb{N}$.

We generally use the following version of the definition above: $(a_n)_{n \in \mathbb{N}}$ is bounded (in $\mathbb{R}$) if, and only if, there exists $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

These two definitions are equivalent because if (a_n) is bounded then $x \leq a_n \leq y$ for all n . Note that $\max\{|y| + 1, |x| + 1\} = R > 0$ and that $y < |y| + 1 \leq R, -R \leq -|x| - 1 < -|x| \leq x$. Therefore $|a_n| \leq R$ for all $n \in \mathbb{N}$. And assuming the alternative definition we get $|a_n| \leq M$ for all $n \in \mathbb{N}$. Then pick $y = M$ and $x = -M$. Done.

Definition 5.4. A sequence of real numbers $(a_n)_{n \in \mathbb{N}}$ is bounded above if, and only if, there exists $y \in \mathbb{R}$ such that $a_n \leq y$ for all $n \in \mathbb{N}$. Similarly (a_n) is bounded below if, and only if, there exists $x \in \mathbb{R}$ such that $x \leq a_n$ for all $n \in \mathbb{N}$.

Proposition 5.1. Every convergent sequence $(a_n)_{n \in \mathbb{N}}$ of real numbers is bounded. Or in terms of its contrapositive if a sequence of real numbers is not bounded it does not converge (diverges).

Proof. For suppose that $\lim_{n \rightarrow \infty} a_n = L \in \mathbb{R}$. Then choosing $\varepsilon = 1$ there exists $n_0 = n_0(1) \in \mathbb{N}$ for which $|a_n - L| < 1$ whenever $n \geq n_0$. But $|a_n| < 1 + |L|$ for every $n \geq n_0$. Now consider $M := \{|a_1|, \dots, |a_{n_0-1}|, 1 + |L|\}$. Then $M > 1$ and then $|a_n| < M$ for all $n \in \mathbb{N}$, which proves that $(a_n)_{n \in \mathbb{N}}$ is bounded (in $\mathbb{R}$). $\square$

Theorem 5.1 (Squeeze Theorem). If $a_n \rightarrow s, c_n \rightarrow s$, and $a_n \leq b_n \leq c_n$ for all but finitely many $n \in \mathbb{N}$, then $b_n \rightarrow s$.

Proof. Let $\varepsilon > 0$ and $I = (s - \varepsilon, s + \varepsilon)$. By assumption, for n sufficiently large, $a_n \in I$. Also, for n sufficiently large, $c_n \in I$. Hence, since I is an interval and b_n lies between a_n and c_n for sufficiently large, we have $b_n \in I$ for sufficiently large. This shows that $b_n \rightarrow s$. $\square$

Theorem 5.2 (Algebraic Limit Theorem). If $a_n \rightarrow a$ and $b_n \rightarrow b$, then:

- (1) $ca_n \rightarrow ca$ for all $c \in \mathbb{R}$.
- (2) $a_n + b_n \rightarrow a + b$.
- (3) $a_n b_n \rightarrow ab$.
- (4) $\frac{a_n}{b_n} \rightarrow \frac{a}{b}$ if $b_n \neq 0$ for all $n \in \mathbb{N}$ and $b \neq 0$.
- (5) $|a_n| \rightarrow |a|$.

Proof. 1. Since $a_n \rightarrow a$, for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $|a_n - a| < \frac{\varepsilon}{|c|}$ (assuming $c \neq 0$). Therefore,

$$|ca_n - ca| = |c||a_n - a| < |c| \cdot \frac{\varepsilon}{|c|} = \varepsilon.$$

This implies $ca_n \rightarrow ca$. If $c = 0$, the result is immediate since $0a_n = 0$ and $0a = 0$.

2. Let $\varepsilon > 0$. Since $a_n \rightarrow a$, there exists N_1 such that for all $n \geq N_1$, $|a_n - a| < \frac{\varepsilon}{2}$. Similarly, since $b_n \rightarrow b$, there exists N_2 such that for all $n \geq N_2$, $|b_n - b| < \frac{\varepsilon}{2}$. For $n \geq \max\{N_1, N_2\}$, we have

$$|(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $a_n + b_n \rightarrow a + b$.

3. Since $a_n \rightarrow a$ and $b_n \rightarrow b$, we know a_n and b_n are bounded. Let $M > 0$ be such that $|a_n| \leq M$ and $|b_n| \leq M$ for all n . Let $\varepsilon > 0$. Since $a_n \rightarrow a$ and $b_n \rightarrow b$, there exist N_1 and N_2 such that for all $n \geq N_1$, $|a_n - a| < \frac{\varepsilon}{2(M+1)}$, and for all $n \geq N_2$, $|b_n - b| < \frac{\varepsilon}{2(M+1)}$. For $n \geq \max\{N_1, N_2\}$,

$$|a_nb_n - ab| = |a_nb_n - a_nb + a_nb - ab| \leq |a_n||b_n - b| + |b||a_n - a|.$$

Since $|a_n| \leq M$ and $|b| \leq M$, we get

$$|a_nb_n - ab| \leq M \cdot \frac{\varepsilon}{2(M+1)} + M \cdot \frac{\varepsilon}{2(M+1)} = \varepsilon.$$

Thus, $a_nb_n \rightarrow ab$.

4. Let $\varepsilon > 0$. Since $b_n \rightarrow b$ and $b \neq 0$, there exists N_1 such that for all $n \geq N_1$, $|b_n - b| < \frac{|b|}{2}$, which implies $|b_n| > \frac{|b|}{2}$. Also, since $a_n \rightarrow a$, there exists N_2 such that for all $n \geq N_2$, $|a_n - a| < \frac{\varepsilon|b|^2}{4}$. For $n \geq \max\{N_1, N_2\}$, we have

$$\left| \frac{a_n}{b_n} - \frac{a}{b} \right| = \left| \frac{a_nb - ab_n}{b_nb} \right| = \left| \frac{a_nb - ab + ab - ab_n}{b_nb} \right| \leq \frac{|b||a_n - a| + |a||b_n - b|}{|b_n||b|}.$$

Since $|b_n| > \frac{|b|}{2}$, we have

$$\left| \frac{a_n}{b_n} - \frac{a}{b} \right| \leq \frac{2}{|b|^2} (|b||a_n - a| + |a||b_n - b|).$$

As $n \rightarrow \infty$, the right-hand side tends to 0, so $\frac{a_n}{b_n} \rightarrow \frac{a}{b}$.

5. By the properties of absolute values and limits, we have

$$||a_n| - |a|| \leq |a_n - a|.$$

Since $a_n \rightarrow a$, the right-hand side tends to 0 as $n \rightarrow \infty$. By the squeeze theorem, $|a_n| \rightarrow |a|$. $\square$

Definition 5.5. A sequence of real numbers (a_n) is said to be

- *monotone increasing* if whenever $n > m$ we have $a_n \geq a_m$;
- *monotone (strictly) increasing* if whenever $n > m$ we have $a_n > a_m$;
- *monotone decreasing* if whenever $n > m$ we have $a_n \leq a_m$;
- *monotone (strictly) decreasing* if whenever $n > m$ we have $a_n < a_m$.

In any of the four cases above (a_n) is called a monotone sequence.

Theorem 5.3 (Monotone Convergence). Any sequence of real numbers (a_n) that is both bounded above and (strictly) monotone increasing converges.

Proof. Since $(a_n)_{n \in \mathbb{N}}$ is bounded there exists $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$ this implies that the set of real numbers $\{a_n : n \in \mathbb{N}\}$ is a bounded subset of $\mathbb{R}$ (in $\mathbb{R}$). Since $\{a_n : n \in \mathbb{N}\} \neq \emptyset$ by the completeness axiom it has a least upper bound/ supremum in $\mathbb{R}$, call it A . For any $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $A - \varepsilon < a_{n_0}$. But $(a_n)_{n \in \mathbb{N}}$ is monotone (strictly) increasing, therefore for any $n > n_0$ we have $A - \varepsilon < a_{n_0} \leq a_n \leq A$ meaning $|a_n - A| < \varepsilon$ for all $n \in \mathbb{N}$ with $n \geq n_0$. This shows that $\lim_{n \rightarrow \infty} a_n = \sup\{a_n : n \in \mathbb{N}\} = \text{lub}\{a_n : n \in \mathbb{N}\}$ whenever (a_n) is bounded and monotone (strictly) increasing. $\square$

Example 5.5. Let $(x_n)_{n \in \mathbb{N}}$ be given by $x_n := \left(1 + \frac{1}{n}\right)^n$ for all $n \in \mathbb{N}$. Prove that it converges and estimate its limit.

By Newton's binomial theorem, we have:

$$\begin{aligned}
 x_n &= 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2} \cdot \left(\frac{1}{n}\right)^2 + \dots + \left(\frac{1}{n}\right)^n \\
 &= 1 + 1 + \frac{1 - \frac{1}{n}}{2!} + \frac{(1 - \frac{1}{n})(1 - \frac{2}{n})}{3!} + \dots + \frac{(1 - \frac{1}{n}) \dots (1 - \frac{i-1}{n})}{i!} + \dots + \left(\frac{1}{n}\right)^n \\
 (1) \quad &= 1 + 1 + \frac{1 - \frac{1}{n}}{2!} + \frac{(1 - \frac{1}{n})(1 - \frac{2}{n})}{3!} + \dots + \frac{(1 - \frac{1}{n}) \dots (1 - \frac{n-1}{n})}{n!}.
 \end{aligned}$$

We know that:

$$x_n \leq 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} = \sum_{i=0}^n \frac{1}{i!}.$$

But $n! \geq 2^{n-1}$ for all $n \in \mathbb{N}$ (this can be proved by induction). Therefore, $\frac{1}{n!} \leq \left(\frac{1}{2}\right)^{n-1}$ for $n \in \mathbb{N}$. Using this inequality, we have:

$$x_n \leq 1 + 1 + \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \dots + \left(\frac{1}{2}\right)^{n-1} = 1 + 2\left(1 - \left(\frac{1}{2}\right)^n\right).$$

Hence:

$$2 < x_n \leq 3 - \left(\frac{1}{2}\right)^{n-1} < 3 \quad \text{for all } n \in \mathbb{N}.$$

This shows (x_n) is bounded.

Now, (x_n) is monotone increasing because, when we compute x_{n+1} , we have:

$$x_{n+1} = \left(1 + \frac{1}{n+1}\right)^{n+1}.$$

Expanding using the binomial theorem:

$$x_{n+1} = 1 + 1 + \frac{1}{2!} \left(1 - \frac{1}{n+1}\right) + \frac{1}{3!} \left(1 - \frac{1}{n+1}\right) \left(1 - \frac{2}{n+1}\right) + \dots + \left(\frac{1}{n+1}\right)^{n+1}.$$

We find that:

$$\left(1 - \frac{1}{n+1}\right) \dots \left(1 - \frac{i-1}{n+1}\right) > \left(1 - \frac{1}{n}\right) \dots \left(1 - \frac{i-1}{n}\right) \quad \text{for } i = 1, 2, \dots, n.$$

Therefore, x_{n+1} has the first n summands greater than x_n and an additional positive term $\left(\frac{1}{n+1}\right)^{n+1}$. This implies $x_{n+1} > x_n$.

By the monotone convergence theorem, $(x_n) = \left(1 + \frac{1}{n}\right)^n$ converges, and its limit lies between 2 and 3.

Definition 5.6 (Euler Number).

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e.$$

Example 5.6. Show that $x_1 = \sqrt{2}, x_2 = \sqrt{2\sqrt{2}}, x_3 = \sqrt{2\sqrt{2\sqrt{2}}}, \dots$ converges and find its limit.

Note that $x_{n+1} := \sqrt{2x_n}$ for all $n \in \mathbb{N}$ and where $x_1 = \sqrt{2}$. For $n = 1$ we have $x_2 = \sqrt{2\sqrt{2}} > x_1 = \sqrt{2}$ because $2\sqrt{2} > 2$. If for certain $n \in \mathbb{N} \setminus \{1\}$ we have $x_{n+1} > x_n$, then $x_{n+1} = \sqrt{2x_n} > x_n$. By definition $x_{n+1}^2 = 2x_n = 2\sqrt{2x_n} > 2x_n$, taking square roots on both sides of the inequality gives $x_{n+2} > \sqrt{2x_n} = x_{n+1}$. By induction we conclude that $x_{n+1} > x_n$ for all $n \in \mathbb{N}$. Then $x_{n+1} = \sqrt{2x_n} > x_n$ for all $n \in \mathbb{N}$, or that $2x_n > x_n^2$. Since $x_n > 0$ for all $n \in \mathbb{N}$ we have $2 > x_n$ for all $n \in \mathbb{N}$, then (x_n) is bounded. By the monotone bounded convergent sequence (x_n) has a limit

l. Then $\lim_{n \rightarrow \infty} x_{n+1} = l = \sqrt{2 \lim_{n \rightarrow \infty} x_n} = \sqrt{2l}$. Solving in terms of l we find either $l = 0$ or $l = 2$. But $x_n > 0$ for all $n \in \mathbb{N}$, meaning $l = 2$.

5.1. Bolzano-Weierstrass Theorem.

Definition 5.7. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of real numbers, and let $(n_k)_{k \in \mathbb{N}}$ be a sequence of natural numbers such that $n_k < n_{k+1}$ for all $k \in \mathbb{N}$. The following sequence of real numbers, defined as

$$a \circ n : \mathbb{N} \longrightarrow \mathbb{R}, \quad a \circ n(k) := a(n(k)) = a(n_k) = a_{n_k},$$

for every $k \in \mathbb{N}$, is called a subsequence of $(a_n)_{n \in \mathbb{N}}$. We denote it by $(a_{n_k})_{k \in \mathbb{N}}$.

Example 5.7. Consider the sequence $(a_n) = ((-1)^n)_{n \in \mathbb{N}}$. Then the sequences

$$(a_{2k})_{k \in \mathbb{N}} = (1)_{k \in \mathbb{N}} \quad \text{and} \quad (a_{2k-1})_{k \in \mathbb{N}} = (-1)_{k \in \mathbb{N}}$$

are subsequences of $((-1)^n)_{n \in \mathbb{N}}$.

Note that we found two convergent subsequences from $((-1)^n)_{n \in \mathbb{N}}$, which is a bounded sequence of real numbers (since $|(-1)^n| = 1$ for any $n \in \mathbb{N}$). Is this property true for any bounded sequence of real numbers?

Example 5.8. Consider the sequence $(a_n)_{n \in \mathbb{N}} = (\sin n)_{n \in \mathbb{N}}$. For all $n \in \mathbb{N}$, we have $|\sin n| \leq 1$. Hence, (a_n) is bounded. With some effort, it can be proven that $(a_n)_{n \in \mathbb{N}}$ does not converge. However, we will see that there exists a convergent subsequence of $(a_n)_{n \in \mathbb{N}}$. It is very difficult to exhibit a convergent subsequence explicitly, but by the Bolzano-Weierstrass theorem, its existence is guaranteed.

Lemma 5.1. Let $(I_n)_{n \in \mathbb{N}}$ be a sequence of closed intervals I_n such that for all $n \in \mathbb{N}$, $I_{n+1} \subseteq I_n$. Then there exists some c such that $c \in I_n$ for all $n \in \mathbb{N}$.

Proof. Let $I_n = [a_n, b_n]$ for all $n \in \mathbb{N}$. Then $I_{n+1} \subseteq I_n$ means that $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$ for all $n \in \mathbb{N}$. Then we have

$$a_1 \leq a_2 \leq \dots \leq a_n \leq b_n \leq \dots \leq b_2 \leq b_1$$

So if $A := \{a_n \in \mathbb{R} : n \in \mathbb{N}\}$ then $A \neq \emptyset$ and A is bounded above by b_1 for example. Then by the completion axiom A has least upper bound, let's call it α . Since $\alpha = \text{lub}(A) = \sup(A)$ it is an upper bound for A , that is $a_n \leq \alpha$ for all $n \in \mathbb{N}$. We know that any b_n is an upper bound for A , but α the least upper bound of A meaning it is less than or equal to any upper bound of A , in particular, $\alpha \leq b_n$ for all $n \in \mathbb{N}$. We have proven that $a_n \leq \alpha \leq b_n$ for all $n \in \mathbb{N}$, that is $\alpha \in [a_n, b_n] = I_n$ for all $n \in \mathbb{N}$. We are done. $\square$

Theorem 5.4 (Bolzano - Weierstrass). Every bounded sequence of real has a convergent subsequence. In other words if $(a_n)_{n \in \mathbb{N}}$ is a bounded sequence of reals (there exists $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$) there exists a sequence of indexes $(n_k)_{k \in \mathbb{N}}$ and a $c \in [-M, M]$ such that $\lim_{k \rightarrow \infty} a_{n_k} = c$.

Proof. Let (a_n) be any bounded sequence of reals. Then there exists $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$. Consider $I_1 := [-M, M]$. Bisecting I_1 we obtain the subintervals: $[-M, 0]$ and $[0, M]$. We claim that at least one of those subintervals I has the following property: for each $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $n \geq k$ and $a_n \in I$; otherwise there would be $k_0 \in \mathbb{N}, \tilde{k}_0 \in \mathbb{N}$ such that for all $n \geq k_0$ we have $a_n \notin [-M, 0]$, and for all $n \geq \tilde{k}_0$ we have $a_n \notin [0, M]$. Then if $n \geq \max\{k_0, \tilde{k}_0\}$ the $a_n \notin [-M, 0]$ or $a_n \notin [0, M]$, which is impossible because $a_n \in [-M, M]$ for all $n \in \mathbb{N}$.

Thus at least one of $[-M, 0]$ or $[0, M]$ have that property. Without loss of generality we can assume that $[0, M]$ has the property above, that is for each $k \in \mathbb{N}$ there exists $n \geq k$ such that $a_n \in [0, M]$. If we consider the subset $\{n \in \mathbb{N} : a_n \in [0, M]\}$ we have that it is non-empty. Therefore by the well-ordering principle in $\mathbb{N}$ there exists the smallest element of $\{n \in \mathbb{N} : a_n \in [0, M]\}$ let's call it n_1 , then $a_{n_1} \in [0, M]$, and we put $I_2 = [0, M]$.

The next step is to bisect I_2 to get two subintervals $[0, \frac{M}{2}]$ and $[\frac{M}{2}, M]$. Again we claim that at least one of those subintervals I have the following property: "for each $k \in \mathbb{N}$ there exists $n \geq k$ such that $a_n \in I$ ". As before if none of the subintervals above did not have the property above we would reach an absurd. Therefore at least one of them have that property. Without loss of generality let's assume it is $[\frac{M}{2}, M]$ (if it was $[0, \frac{M}{2}]$ the argument is the same) with that property. Then for all $k \in \mathbb{N}$ there exists $n \geq k$ such that $a_n \in [\frac{M}{2}, M]$. In particular we can choose $k = n_1 + 1$ then there exists $n \geq n_1 + 1$ such that $a_n \in [\frac{M}{2}, M]$. Then the subset of naturals $\{n \in \mathbb{N} : n \geq n_1 + 1 \text{ and } a_n \in [\frac{M}{2}, M]\}$ is non-empty and by the well-ordering principle there exists the smallest element of this subset which is n_2 . Then $a_{n_2} \in [\frac{M}{2}, M]$ and we put $I_3 = [\frac{M}{2}, M]$.

Continuing this way we find for all $k > 1$ a subinterval $I_{k+1} \subseteq I_k$ where I_k is obtained in the previous step such that:

- (1) for $n_k := \min\{n \in \mathbb{N} : n \geq n_{k-1} + 1 \text{ and } a_n \in I_{k+1}\}$;
- (2) The length of I_{k+1} is half of the length of I_k .

This tells us that there exists a sequence of points $(a_k) \in \mathbb{I}_{k+1}$ for $k \in \mathbb{N}$ and a square of closed intervals obtained by bisecting the previous ones.

By Lemma 1 (nested interval theorem) there exists $c \in I_k$ for all $k \in \mathbb{N}$ and since the lengths of $(I_k)_{k \in \mathbb{N}}$ are getting smaller this c is unique. Now we claim that $\lim_{k \rightarrow \infty} a_{n_k} = c$. In fact since $a_{n_k} \in I_{k+1}$ and $c \in I_k$ for all $k \in \mathbb{N}$ and $n \in \mathbb{N}$ we have that $a_{n_k} \in I_{k+1}$ and $c \in I_{k+1}$ for all $k \in \mathbb{N}$. But the fact that $a_{n_k}, c \in I_{k+1}$ for all $k \in \mathbb{N}$ means that $|a_{n_k} - c| \leq l(I_{k+1}) = \text{length of } I_{k+1} = \frac{l(I_1)}{2^k} = \frac{M}{2^{k-1}}$ for every $k \in \mathbb{N}$. Since $\lim_{k \rightarrow \infty} \frac{M}{2^{k-1}} = 0$ then $\lim_{k \rightarrow \infty} |a_{n_k} - c| = 0$. $\square$

Definition 5.8. A sequence $(a_n)_{n \in \mathbb{N}}$ of real numbers is a Cauchy sequence if, and only if, for every $\varepsilon > 0$ there exists $N_0 = N_0(\varepsilon) \in \mathbb{N}$ such that $|a_m - a_n| < \varepsilon$ for all $m, n \in \mathbb{N}$ such that $m, n \geq N_0$.

Theorem 5.5 (Cauchy Criterion). A sequence of real numbers converges if, and only if, it is a Cauchy sequence.

Problem 5.1 (3.11). Using only the definition of the limit, show that $\lim_{n \rightarrow \infty} \frac{3n-1}{n+2} = 3$

Solution 5.1. $a_n - 3 = \frac{3n-1}{n+2} - 3 = \frac{-7}{n+2} \rightarrow |a_n - 3| = \frac{7}{n+2}$ We must show that for some $n \geq n_0$ we have $|a_n - 3| < \varepsilon$ for all $\varepsilon > 0$.

$$\frac{7}{n+2} < \varepsilon$$

If we choose $n_0 > \frac{7}{\varepsilon} - 2$ for all $n \geq n_0$ we will have $\frac{7}{n+2} < \varepsilon$ for all $\varepsilon > 0$. Thus, $\lim_{n \rightarrow \infty} \frac{3n-1}{n+2} = 3$.

Problem 5.2 (3.12). Let (a_n) be a sequence. Define a sequence (b_n) by the formula $b_n = a_{n+1}$ for each $n \in \mathbb{N}$. In other words, (b_n) is (a_n) with the first term removed. Prove that $a_n \rightarrow c$ if and only if $b_n \rightarrow c$.

Solution 5.2. ($\Rightarrow$) Assume $a_n \rightarrow c$. So for every $\varepsilon > 0$, there is some n_0 such that for all $n \geq n_0$

$$|a_n - c| < \varepsilon$$

For $b_n = a_{n+1}$, as $n+1 > n \geq n_0$, we have $|b_n - c| = |a_{n+1} - c| < \varepsilon$. Thus, we arrived at $b_n \rightarrow c$ from $a_n \rightarrow c$.

($\Leftarrow$) Assume $b_n \rightarrow c$. For every $\varepsilon > 0$, there exists n_0 such that for all $n \geq n_0$

$$|b_n - c| = |a_{n+1} - c| < \varepsilon$$

$n+1 > n \geq n_0$ we have $a_n \rightarrow c$, from $b_n \rightarrow c$.

Problem 5.3 (3.13). Prove the following statement or disprove it by a counterexample. If (a_n) and (b_n) are sequences such that (a_n) and $(a_n b_n)$ converge, then (b_n) converges.

Solution 5.3. Suppose $a_n \rightarrow a$, $a_n b_n \rightarrow c$. We must show that $b_n \rightarrow b$ for some b .

Take $a_n = \frac{1}{n}$, $b_n = n$. We know $a_n b_n \rightarrow 1$, $a_n \rightarrow 0$. However, $b_n = n$ diverges.

Problem 5.4 (3.14). Prove the following statement or disprove it by a counterexample. If (a_n) and (b_n) are sequences of positive numbers such that (a_n) and (a_n/b_n) converge, then (b_n) converges.

Solution 5.4. Let $a_n \rightarrow A$, and $\frac{a_n}{b_n} \rightarrow C$ for $A, C > 0$.

$$b_n = \frac{a_n}{\frac{a_n}{b_n}}$$

When $n \rightarrow \infty$, from $a_n \rightarrow A$, and $\frac{a_n}{b_n} \rightarrow C$ we will have $b_n \rightarrow \frac{A}{C}$.

Problem 5.5 (3.15). Let (a_n) be a bounded (not necessarily convergent) sequence and (b_n) be a sequence such that $b_n \rightarrow 0$. Show that $a_n b_n \rightarrow 0$.

Solution 5.5. Since (a_n) is bounded, there will be a $M > 0$ such that $|a_n| \leq M$ for all n . $b_n \rightarrow 0$, for every $\varepsilon > 0$ there is n_0 such that for all $n \geq n_0$ we have $|b_n| < \frac{\varepsilon}{M}$.

Thus for $n \geq n_0$,

$$|a_n b_n| \leq |a_n| |b_n| \leq M |b_n| < M \cdot \frac{\varepsilon}{M} = \varepsilon$$

Thus, $a_n b_n \rightarrow 0$.

Problem 5.6. • Let $a \geq 0$ and $n \in \mathbb{N}$. Show that $(1+a)^n \geq na$.

• Let $c > 0$. Show that $c^{\frac{1}{n}} \rightarrow 1$ as $n \rightarrow \infty$.

Solution 5.6. • We will prove by induction on n .

Base case $n = 1$, $(1+a)^1 = 1+a \geq 1a = a$.

Assume that our hypothesis holds on k , $(1+a)^k \geq ka$. We will prove that it holds for $n = k+1$.

$$(1+a)^{k+1} = (1+a)^k \cdot (1+a) \geq ka \cdot (1+a)$$

As $k(1+a) \geq k+1$ by $1+a \geq 1$, we deduce that $ak(1+a) \geq a(k+1)$. Thus, $(1+a)^{k+1} \geq a(k+1)$.

- Let $c > 1$. Denote $a = c^{\frac{1}{n}} - 1$. It is equivalent with $c = (1 + a)^n$, and from Binomial equation,

$$c = (1 + a)^n = 1 + na + \frac{n(n-1)}{2}a^2 + \dots$$

$$\frac{c-1}{n} = a + \frac{n-1}{2}a^2 + \dots$$

As n approaches ∞ , $LHS \rightarrow 0 \rightarrow RHS \rightarrow a \rightarrow 0$. So, $c^{\frac{1}{n}} - 1 \rightarrow 0$. Conversely, $c^{\frac{1}{n}} \rightarrow 1$.

6. SOME TOPOLOGY OF THE REAL LINE

Definition 6.1. Let $X \subseteq \mathbb{R}$. We say that X is an open set in $(\mathbb{R}, |\cdot|)$ (the real line) if, and only if, for every $x \in X$ there exists $r = r(x) > 0$ such that the open interval $(x - r, x + r) \subseteq X$.

Example 6.1. $X = \emptyset$ and $X = \mathbb{R}$ are open sets of the real line. Note that if $X \subsetneq \mathbb{R}$ was not an open set of the real line, then there would be some $x_0 \in \mathbb{R}$ such that for all $r > 0$ the $(x_0 - r, x_0 + r) \not\subseteq X$. But $x_0 \in X$ is an absurd. Hence X is always an open set.

For $X = \mathbb{R} \subseteq \mathbb{R}$ given any $x \in \mathbb{R}$ and any $r > 0$ we have $(x - r, x + r) \subseteq \mathbb{R} = X$. Thus $X = \mathbb{R}$ is an open set of the real line.

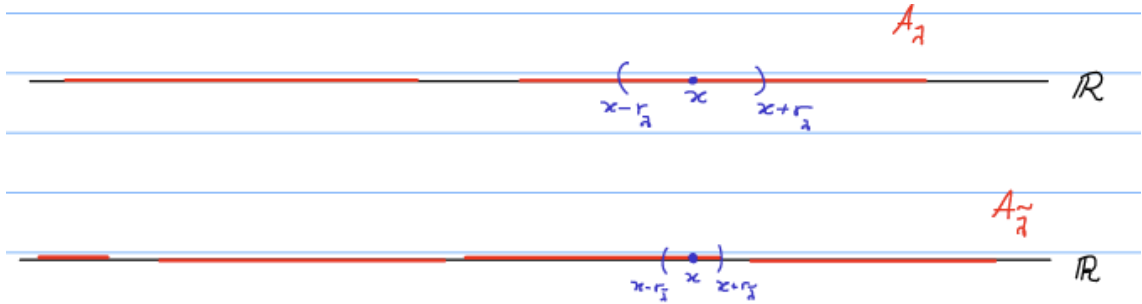
Problem 6.1. Prove that any open interval is an open set of the real line.

Proposition 6.1. Let $(A_\lambda)_{\lambda \in I}$ be a family of open sets of the real line indexed by I . Then

- $\bigcup_{\lambda \in I} A_\lambda$ is also an open set of the real line
- If $I = \{1, \dots, n\}$, then $\bigcap_{\lambda \in I} A_\lambda$ is an open set. In general, $\bigcap_{\lambda \in I} A_\lambda$ is not an open set if I is arbitrary.

Proof. • Pick any $x \in \bigcup_{\lambda \in I} A_\lambda$, then there exists $\lambda \in I$ such that $x \in A_\lambda$ is an open set of $\mathbb{R}$, there exists, by definition, some $r = r(x, \lambda) > 0$ such that $(x - r, x + r) \subseteq A_\lambda$. But $A_\lambda \subseteq \bigcup_{\lambda \in I} A_\lambda$ therefore $(x - r, x + r) \subseteq \bigcup_{\lambda \in I} A_\lambda$. Thus, $\bigcup_{\lambda \in I} A_\lambda$ is an open set of $\mathbb{R}$.

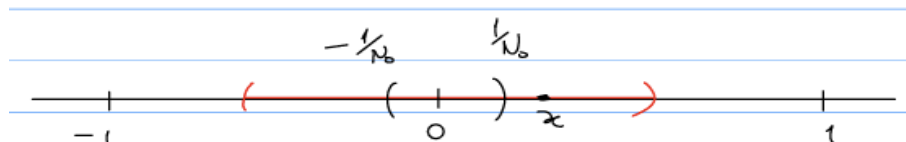
• Pick any $x \in \bigcap_{\lambda \in I} A_\lambda$, therefore $x \in A_\lambda$ for all $\lambda \in I = \{1, \dots, n\}$. Since A_λ is an open set of $\mathbb{R}$ there exists $r_\lambda > 0$ such that $(x - r_\lambda, x + r_\lambda) \subseteq A_\lambda$ for $\lambda \in \{1, \dots, n\}$



The drawing above shows the dependency on r for each $\lambda \in \{1, \dots, n\}$. So, to construct an open interval centered at x that will be a subset of $\bigcap_{\lambda \in I} A_\lambda$, we pick the smallest of all $r = r_\lambda$ for $\lambda \in \{1, \dots, n\}$, that is, we define $\underline{r} = \min\{r_\lambda \mid \lambda \in \{1, \dots, n\}\} = \min\{r_1, \dots, r_n\} > 0$ and we have $\underline{r} \leq r_\lambda$ for all $\lambda \in I = \{1, \dots, n\}$. This guarantees that $(x - \underline{r}, x + \underline{r}) \subseteq (x - r_\lambda, x + r_\lambda)$ for all $\lambda \in I = \{1, \dots, n\}$, in other words $(x - \underline{r}, x + \underline{r}) \subseteq \bigcap_{\lambda \in I} A_\lambda$. Then $\bigcap_{\lambda \in I} A_\lambda$ is an open set of $\mathbb{R}$.

□

Consider the following sequence of open intervals $\left(\left(-\frac{1}{n}, \frac{1}{n}\right)\right)_{n \in \mathbb{N}}$. By the exercise above $\left(-\frac{1}{n}, \frac{1}{n}\right) \subseteq \mathbb{R}$ are open sets for each $n \in \mathbb{N}$. But $\bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$ because $0 \in \left(-\frac{1}{n}, \frac{1}{n}\right)$ for all $n \in \mathbb{N}$ and if there exists $x \in \bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right)$ then $-\frac{1}{n} < x < \frac{1}{n}$ for all $n \in \mathbb{N}$. Note that if we had $x \neq 0$ there would be a $N_0 \in \mathbb{N}$ so large that $|x| \geq \frac{1}{N_0}$, but in this case $x \notin \left(-\frac{1}{n}, \frac{1}{n}\right)$ for $n \in \mathbb{N}$ such that $n \geq N_0$. This is impossible because we picked $x \in \bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right)$



Thus, $x = 0$.

Notice that for each $r > 0$ the $(-r, r) = (0 - r, 0 + r) \not\subseteq \{0\} = \bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right)$ hence $\{0\}$ cannot be an open set.

Definition 6.2. Let $\mathbb{F} \in \mathbb{R}$. We say that $\mathbb{F}$ is a closed set of the real line if and only if $\mathbb{F}^c = \mathbb{R} - \mathbb{F}$ is an open set of the real line.

Problem 6.2. Prove that any closed interval is a closed set of the real line.

Proposition 6.2. Let $(A_\lambda)_{\lambda \in I}$ be a family of closed sets of the real line indexed by I . Then:

- (i) $\bigcap_{\lambda \in I} A_\lambda$ is also a closed set of the real line.
- (ii) If $I = \{1, \dots, n\}$, then $\bigcup_{\lambda \in I} A_\lambda$ is a closed set. In general, the $\bigcup_{\lambda \in I} A_\lambda$ is *not* a closed set if I is arbitrary.

Proof. We invoke the De Morgan laws which tell us that the following set theoretical identities hold:

$$\begin{aligned} \left(\bigcap_{\lambda \in I} A_\lambda\right)^c &= \mathbb{R} - \bigcap_{\lambda \in I} A_\lambda = \bigcup_{\lambda \in I} A_\lambda^c = \bigcup_{\lambda \in I} (\mathbb{R} - A_\lambda) \\ \left(\bigcup_{\lambda \in I} A_\lambda\right)^c &= \mathbb{R} - \bigcup_{\lambda \in I} A_\lambda = \bigcap_{\lambda \in I} A_\lambda^c = \bigcap_{\lambda \in I} (\mathbb{R} - A_\lambda) \end{aligned}$$

Since A_λ is a closed set of $\mathbb{R}$ for all $\lambda \in I$, by definition, $A_\lambda^c = \mathbb{R} - A_\lambda$ is an open set of $\mathbb{R}$. From Proposition 1, we find that:

$$\bigcup_{\lambda \in I} A_\lambda^c = \left(\bigcap_{\lambda \in I} A_\lambda\right)^c$$

is an open set. Therefore, $\bigcap_{\lambda \in I} A_\lambda$ is a closed set of $\mathbb{R}$.

Now, if $I = \{1, \dots, n\}$, then $\bigcap_{\lambda \in I} A_\lambda^c$ is also an open set of $\mathbb{R}$, thanks to Proposition 1 above. But we know that:

$$\left(\bigcup_{\lambda \in I} A_\lambda\right)^c = \bigcap_{\lambda \in I} A_\lambda^c$$

therefore $\bigcup_{\lambda \in I} A_\lambda$ is a closed set of $\mathbb{R}$.

If I is arbitrary, the union does not need to be a closed set. In fact, if one considers the family/sequence $\left(-1 + \frac{1}{n}, 1 - \frac{1}{n}\right)$ indexed by $\mathbb{N}$, we see that it is a family of closed sets of $\mathbb{R}$ (by the exercise above) and:

$$\bigcup_{n \in \mathbb{N}} \left[-1 + \frac{1}{n}, 1 - \frac{1}{n}\right] = (-1, 1),$$

which is *not* a closed set. □

Theorem 6.1 (Sequential Characterization of Closed Sets of $\mathbb{R}$). Let $\mathbb{F} \subseteq \mathbb{R}$. $\mathbb{F}$ is a closed set of $\mathbb{R}$ if, and only if, for every sequence $(a_n)_{n \in \mathbb{N}}$ such that $a_n \in \mathbb{F}$ for all $n \in \mathbb{N}$, and $\lim_{n \rightarrow \infty} a_n = c$ exists we have that $c \in \mathbb{F}$. For short: $\mathbb{F}$ is closed if, and only if, every convergent sequence defined in $\mathbb{F}$ its limit is also in $\mathbb{F}$.

Proof. ($\implies$) Suppose $\mathbb{F}$ is a closed set of $\mathbb{R}$, and let (a_n) be any convergent sequence defined in $\mathbb{F}$ and $c = \lim_{n \rightarrow \infty} a_n$. We claim that $c \in \mathbb{F}$. If it wasn't the case then $c \in \mathbb{R} - \mathbb{F}$. Since $\mathbb{F}$ is a closed set, by definition, $\mathbb{F}^c = \mathbb{R} - \mathbb{F}$ is an open set of $\mathbb{R}$ therefore there exists some $r_0 > 0$ such that the open interval $(c - r_0, c + r_0) \subseteq \mathbb{F}^c = \mathbb{R} - \mathbb{F}$. □



But $(a_n)_{n \in \mathbb{N}}$ converges to c and $a_n \in \mathbb{F}$ for all n . Then for $\varepsilon = r_0 > 0$ there would be $n_0 \in \mathbb{N}$ such that $|a_n - c| < r_0$ for all $n \geq n_0$. But $(c - r_0, c + r_0)$ contains no point from $\mathbb{F}$, which contradicts the convergence of (a_n) to c . Then we conclude that $c \in \mathbb{F}$.

($\Leftarrow$) If for every convergent sequence $(a_n)_{n \in \mathbb{N}}$ such that $a_n \in \mathbb{F}$ for all $n \in \mathbb{N}$ and $c = \lim_{n \rightarrow \infty} a_n \in \mathbb{F}$, we have that for every $X \in \mathbb{F}^c = \mathbb{R} - \mathbb{F}$ there exists $r_0 > 0$ such that $(x - r_0, x + r_0) \subseteq \mathbb{F}^c = \mathbb{R} - \mathbb{F}$. If this was false, then there is $x \in \mathbb{F}^c$ such that for every $r > 0$ we have that $(x - r, x + r) \cap \mathbb{F} \neq \emptyset$. For each $n \in \mathbb{N}$ let $r = \frac{1}{n}$. From the fact that $(x - \frac{1}{n}, x + \frac{1}{n}) \cap \mathbb{F} \neq \emptyset$ we can pick some $x_n \in \mathbb{F}$ such that $|x_n - x| < \frac{1}{n}$ for all $n \in \mathbb{N}$. Therefore we constructed a sequence $(x_n)_{n \in \mathbb{N}}$ with $x_n \in \mathbb{F}$ for all $n \in \mathbb{N}$. Therefore we constructed a sequence $(x_n)_{n \in \mathbb{N}}$ with $x_n \in \mathbb{F}$ for all $n \in \mathbb{N}$ and that $\lim_{n \rightarrow \infty} x_n = x$, but $x \in \mathbb{F}^c = \mathbb{R} - \mathbb{F}$, which contradicts the first assumption above. This guarantees that for every $x \in \mathbb{F}^c$, for some $r_0 > 0$ we have $(x - r_0, x + r_0) \subseteq \mathbb{F}^c$, that $\mathbb{F}^c = \mathbb{R} - \mathbb{F}$ is an open set of $\mathbb{R}$.

Definition 6.3. A set $\mathbb{K} \subseteq \mathbb{R}$ is a *compact set* of the real line if, and only if, for every sequence $(a_n)_{n \in \mathbb{N}}$ in $\mathbb{K}$ (that is $a_n \in \mathbb{K}$ for each $n \in \mathbb{N}$) there exists a convergent subsequence of $(a_n)_{n \in \mathbb{N}}$, say $(a_{n_j})_{j \in \mathbb{N}}$, such that $L = \lim_{j \rightarrow \infty} a_{n_j} \in \mathbb{K}$.

Example 6.2. Any closed interval of $\mathbb{R}$ is a compact set. Here $\mathbb{K} = [a, b]$, and let $(a_n)_{n \in \mathbb{N}}$ be any sequence in $[a, b]$. Then (a_n) is a bounded sequence of reals. By Bolzano-Weierstrass theorem (a_n) has a convergent subsequence (a_{n_j}) whose limit $\lim_{j \rightarrow \infty} a_{n_j} \in [a, b]$.

Example 6.3. $\mathbb{K} := \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ is a compact set of $\mathbb{R}$.

Theorem 6.2 (Heine-Borel on $\mathbb{R}$). A subset $\mathbb{K}$ of $\mathbb{R}$ is compact set of $\mathbb{R}$ if, and only if, $\mathbb{K}$ is a closed and bounded set of $\mathbb{R}$.

Proof. ($\Rightarrow$) Let $\mathbb{K} \subseteq \mathbb{R}$ be a compact set of $\mathbb{R}$, then any sequence of points in $\mathbb{K}$ has a convergent subsequence and its limit belongs to $\mathbb{K}$. We claim that $\mathbb{K}$ is a closed set of $\mathbb{R}$. In fact, for any convergent sequence $(a_n)_{n \in \mathbb{N}}$ in $\mathbb{K}$ we have that its limit belongs to $\mathbb{K}$ because there exists a subsequence $(a_{n_j})_{j \in \mathbb{N}}$ of (a_n) , which converges (remember $\mathbb{K}$ is compact) to some element of $\mathbb{K}$, say $c = \lim a_{n_j}$. But (a_n) is convergent, therefore all of its subsequences converge and their limits are the same, that is $\lim_{n \rightarrow \infty} a_n = \lim_{j \rightarrow \infty} a_{n_j} = c \in \mathbb{K}$. Thus any convergent sequence in $\mathbb{K}$ has its limit in $\mathbb{K}$, thanks to the sequence characterization of closed sets above we conclude that $\mathbb{K}$ is a closed set of $\mathbb{R}$.

Note that $\mathbb{K}$ is also a bounded set. If $\mathbb{K}$ wasn't a bounded set then for every $n \in \mathbb{N}$ we have that $\mathbb{K} \not\subseteq [-n, n]$, that is there exists some element $\alpha_n \in \mathbb{K}$ such that $|\alpha_n| > n$. So we constructed a sequence $(\alpha_n)_{n \in \mathbb{N}}$ in $\mathbb{K}$ such that for all $n \in \mathbb{N}$ we have $|\alpha_n| > n$, that is, $(\alpha_n)_{n \in \mathbb{N}}$ is not a bounded sequence therefore it cannot converge. But $\mathbb{K}$ is compact, then there exists a subsequence $(\alpha_{n_j})_{j \in \mathbb{N}}$ of $(\alpha_n)_{n \in \mathbb{N}}$ which converges to some element of $\mathbb{K}$. This is impossible because $|\alpha_{n_j}| > n_j$ for all $j \in \mathbb{N}$, meaning $(\alpha_{n_j})_{j \in \mathbb{N}}$ is not bounded. We conclude that $\mathbb{K}$ is bounded.

($\Leftarrow$) For suppose that $\mathbb{K}$ is closed and bounded, Let $(x_n)_{n \in \mathbb{N}}$ be any sequence in $\mathbb{K}$. Since $\mathbb{K}$ is bounded there exists $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$, meaning $(x_n)_{n \in \mathbb{N}}$ is a bounded sequence in $\mathbb{K}$. By Bolzano-Weierstrass theorem there is a convergent subsequence $(x_{n_j})_{j \in \mathbb{N}}$ if $(x_n)_{n \in \mathbb{N}}$ and its limit $\tilde{x} = \lim_{j \rightarrow \infty} x_{n_j}$ belongs to $[-M, M]$. Since $\mathbb{K}$ is closed by the sequence characterization of closed sets of $\mathbb{R}$ we have that for $(x_{n_j})_{j \in \mathbb{N}}$ it converges to $\tilde{x}$ and $\tilde{x} \in \mathbb{K}$. Thus we have proven that every sequence in $\mathbb{K}$ has a convergent subsequence and its limit belongs to $\mathbb{K}$, that is, we have proven $\mathbb{K}$ is compact. $\square$

7. LIMITS OF FUNCTIONS AND CONTINUITY

7.1. Limits of real valued functions of one real variable.

Definition 7.1. Let $A \subseteq \mathbb{R}$, and $x_0 \in \mathbb{R}$. We say that x_0 is a limit point of the set A if, and only if, there exists a sequence $(a_n)_{n \in \mathbb{N}}$ in A such that $a_n \neq x_0$ and $\lim_{n \rightarrow \infty} a_n = x_0$. Or equivalently, x_0 is a limit point of A if, and only if, for all $\varepsilon > 0$ we have $((a - \varepsilon, a + \varepsilon) \setminus \{a\}) \cap A \neq \emptyset$. The set of all limit points of A is A' , called the derived set of A and $A' \subseteq \mathbb{R}$.

Example 7.1. Let $A = \mathbb{Q}$. We claim that the set of all limit points of A is $\mathbb{R}$, that is $\mathbb{Q}' = \mathbb{R}$.

Note that for any real x_0 and $n \in \mathbb{N}$ in $(x_0 - \frac{1}{n}, x_0 + \frac{1}{n})$ there are rationals ($\mathbb{Q}$ is dense in $\mathbb{R}$), let's choose $r_n \in \mathbb{Q} \cap (x_0 - \frac{1}{n}, x_0 + \frac{1}{n})$ for $n = 1, 2, \dots$. Then we have a sequence of rationals $(r_n)_{n \in \mathbb{N}}$ such that $|r_n - x_0| < \frac{1}{n}$ for all $n \in \mathbb{N}$, that is (r_n) converges to x_0 . Then x_0 is a limit point of $A = \mathbb{Q}$. And this shows that $\mathbb{R} \subseteq \mathbb{Q}' = \text{set of all limit points of } \mathbb{Q}$. Since $\mathbb{Q}' \subseteq \mathbb{R}$ is immediate.

Example 7.2. Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. Then $x_0 = 0$ is a limit point of A , but $x_0 = \frac{1}{\sqrt{2}}$ is not a limit point.

Note that for $(a_n)_{n \in \mathbb{N}} := (\frac{1}{n})_{n \in \mathbb{N}}$ we have (a_n) is a sequence in A and $\lim_{n \rightarrow \infty} a_n = 0$. Then $x_0 = 0$ is a limit of $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. For suppose that $x_0 = \frac{1}{\sqrt{2}}$ is a limit point of $A = \{\frac{1}{n} : n \in \mathbb{N}\}$, then there exists a sequence of points in A , say $(a_n)_{n \in \mathbb{N}}$ which converges to $\frac{1}{\sqrt{2}} = x_0$. Then for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|a_n - \frac{1}{\sqrt{2}}| < \varepsilon$ for all $n \geq n_0$. But if we choose $\varepsilon = \min\{1 - \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} - \frac{1}{2}\} > 0$, then for certain N_0 we have that $|a_n - \frac{1}{\sqrt{2}}| < \varepsilon$ for all $n \geq n_0$. But $(\frac{1}{\sqrt{2}} - \varepsilon, \frac{1}{\sqrt{2}} + \varepsilon) \cap A = \emptyset$ which contradicts the existence of (a_n) converging to $\frac{1}{\sqrt{2}}$. Then $x_0 = 1$ cannot be a limit point of $A = \{\frac{1}{n} : n \in \mathbb{N}\}$.

Definition 7.2 (Limit of a Function - Sequential Limits of Functions). Let $\mathbb{X} \subseteq \mathbb{R}$, $x_0 \in \mathbb{X}'$ and $f : \mathbb{X} \rightarrow \mathbb{R}$ be a function (which may or may not be defined at x_0). We say that $L \in \mathbb{R}$ is a limit of f at x_0 (as x tends to x_0) if, and only if, for every sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathbb{X}$ which converges to x_0 we have that $\lim_{n \rightarrow +\infty} f(x_n) = L$.

$\tilde{L} \in \mathbb{R}$ is not a limit of f at x_0 (as x tends to x_0) if, and only if, there exists a sequence $(\tilde{x}_n)_{n \in \mathbb{N}}$ in $\mathbb{X}$ which converges to x_0 but the $\lim_{n \rightarrow \infty} f(\tilde{x}_n) \neq \tilde{L}$.

Since the limit of sequences is unique

Example 7.3. Let $f : [0, 1] \rightarrow \mathbb{R}$ be given by $f(x) = \frac{x}{x^2+1}$. We want to determine if $\lim_{x \rightarrow \frac{1}{5}} f(x)$ exists or it doesn't exist.

Let (x_n) be any sequence on $[0, 1]$ converging to $x_0 = \frac{1}{5}$, that is, $\lim_{n \rightarrow \infty} x_n = \frac{1}{5}$ and $x_n \in [0, 1]$ for all $n \in \mathbb{N}$. Notice that $f(x_n) = \frac{x_n}{x_n^2+1}$ then by the limit laws for sequences we have

$$\lim_{n \rightarrow +\infty} f(x_n) = \frac{\lim_{n \rightarrow \infty} x_n}{1 + \left(\lim_{n \rightarrow \infty} x_n\right)^2} = \frac{\frac{1}{5}}{1 + \frac{1}{25}} = \frac{5}{26}$$

Therefore $\lim_{x \rightarrow \frac{1}{5}} f(x) = \frac{5}{26}$

Problem 7.1. Does $\lim_{x \rightarrow \frac{1}{5}} \frac{1}{x^2}$ exist? Give reasons

Solution 7.1. The function $\frac{1}{x^2}$ is defined and continuous for all real numbers except $x = 0$. As $x \rightarrow \frac{1}{5}$,

$$x^2 \rightarrow \left(\frac{1}{5}\right)^2 = \frac{1}{25}$$

Thus,

$$\frac{1}{x^2} \rightarrow \frac{1}{\frac{1}{25}} = 25$$

Therefore,

$$\lim_{x \rightarrow \frac{1}{5}} \frac{1}{x^2} = 25$$

Problem 7.2. Does $\lim_{x \rightarrow 1} \frac{x^m - 1}{x - 1}$ exists ?

Solution 7.2. Using the difference of powers formula:

$$x^m - 1 = (x - 1)(x^{m-1} + x^{m-2} + \cdots + x + 1)$$

$$= \lim_{x \rightarrow 1} (x^{m-1} + x^{m-2} + \cdots + x + 1)$$

As $x \rightarrow 1$, every term in the sum becomes 1:

$$= 1 + 1 + \cdots + 1 \quad (m \text{ terms}) = m$$

Thus,

$$\lim_{x \rightarrow 1} \frac{x^m - 1}{x - 1} = m$$

Definition 7.3 ($\varepsilon - \delta$ form of limit). Let $X \subseteq \mathbb{R}$, $x_0 \in X'$, and $f : X \rightarrow \mathbb{R}$ be a function (which may or may not be defined at x_0). We say that $L \in \mathbb{R}$ is a limit of f at x_0 (as x tends to x_0) if, and only if, for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, x_0, f) > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{for all } x \in X \text{ such that } 0 < |x - x_0| < \delta.$$

We can prove that these two definitions are equivalent. That is, if we assume either one of them we get the other one.

Proposition 7.1. Let $X \subseteq \mathbb{R}$, $x_0 \in X'$, $f : X \rightarrow \mathbb{R}$ be a function, and $L \in \mathbb{R}$ be a limit of f at x_0 . If $\tilde{L} \in \mathbb{R}$ is also a limit of f at x_0 , then $L = \tilde{L}$.

Proof. By definition, given $\varepsilon > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that:

- (i) $|f(x) - L| < \frac{\varepsilon}{2}$ for all $x \in X$ such that $0 < |x - x_0| < \delta_1$,
- (ii) $|f(x) - \tilde{L}| < \frac{\varepsilon}{2}$ for all $x \in X$ such that $0 < |x - x_0| < \delta_2$.

For $\delta = \min\{\delta_1, \delta_2\} > 0$, we have $|f(x) - L| < \frac{\varepsilon}{2}$ and $|f(x) - \tilde{L}| < \frac{\varepsilon}{2}$ for all $x \in X$ such that $0 < |x - x_0| < \delta$. Then, using the triangle inequality, we have

$$|L - \tilde{L}| \leq |L - f(x)| + |f(x) - \tilde{L}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $x \in X$ such that $0 < |x - x_0| < \delta$. Since $\varepsilon > 0$ was arbitrary, our deductions led to $|L - \tilde{L}| < \varepsilon$. The only way for it to be true is if $|L - \tilde{L}| = 0$, that is, $L = \tilde{L}$.

Hence, the limit is unique. $\square$

Proposition 7.2 (Limit Laws). Let $f, g : X \rightarrow \mathbb{R}$ such that $X \subseteq \mathbb{R}$, x_0 is a limit point of x . For suppose that $\lim_{x \rightarrow x_0} f(x) = L$ and $\lim_{x \rightarrow x_0} g(x) = \tilde{L}$. Then the following hold

$$(1) \lim_{x \rightarrow x_0} (f + g)(x) = L + \tilde{L}$$

- (2) $\lim_{x \rightarrow x_0} (f \cdot g)(x) = L \cdot \tilde{L}$
 (3) If $L \neq 0$ then $\lim_{x \rightarrow x_0} \frac{1}{f(x)} = \frac{1}{L}$
 (4) $\lim_{x \rightarrow x_0} |f(x)| = |L|$

Proof. For the proof we just inverse the theorem on limit laws for sequences of real numbers. In fact, for every sequence (x_n) in x such that $\lim_{n \rightarrow \infty} x_n = x_0$ then we consider the sequence $(f(x_n))_{n \geq 1}$. This last sequence converges because $\lim_{x \rightarrow x_0} f(x) = L$ and $\lim_{x \rightarrow x_0} g(x) = \tilde{L}$ exist. Then $\lim_{n \rightarrow \infty} (f + g)(x_n) = \lim_{n \rightarrow \infty} (f(x_n) + g(x_n)) = \lim_{n \rightarrow \infty} f(x_n) + \lim_{n \rightarrow \infty} g(x_n) = L + \tilde{L}$. In other words

$$\lim_{x \rightarrow x_0} (f + g)(x) = L + \tilde{L}$$

Similarly for $\lim_{x \rightarrow x_0} f(x)g(x)$ we have that the sequence $(f(x_n) \cdot g(x_n))_{n \geq 1}$ converges. By the limit laws for sequences discussed previously, the limit of $f(x_n)g(x_n)$ must be

$$\lim_{n \rightarrow \infty} f(x_n)g(x_n) = \lim_{n \rightarrow \infty} f(x_n) \lim_{n \rightarrow \infty} g(x_n) = L \cdot \tilde{L}$$

For the 3rd property, as we have $\lim_{x \rightarrow x_0} f(x) = L$, we have some $\delta_1 > 0$ such that $|f(x) - L| < \frac{|L|}{2}$, ($0 < |x - x_0| < \delta_1$). From triangle inequality we know that

$$|L| = |L - f(x) + f(x)| \leq |L - f(x)| + |f(x)| = |f(x) - L| + |f(x)| < \frac{|L|}{2} + |f(x)|$$

Thus,

$$\frac{|L|}{2} < |f(x)| \longrightarrow \frac{1}{|f(x)|} < \frac{2}{|L|}$$

We also have for some $\delta_2 > 0$, $|f(x) - L| < \frac{|L|^2}{2}\varepsilon$, ($0 < |x - x_0| < \delta_2$). Now we choose $\delta = \min\{\delta_1, \delta_2\}$. As $0 < |x - x_0| < \delta$, we have.

$$\left| \frac{1}{f(x)} - \frac{1}{L} \right| = \frac{1}{|Lf(x)|} |L - f(x)| = \frac{1}{|L|} \frac{1}{|f(x)|} |f(x) - L| < \frac{1}{|L|} \frac{2}{|L|} |f(x) - L| < \frac{2}{|L|^2} \frac{|L|^2}{2} \varepsilon = \varepsilon$$

Thus, if $L \neq 0$ then $\lim_{x \rightarrow x_0} \frac{1}{f(x)} = \frac{1}{L}$.

As we have $\lim_{x \rightarrow x_0} f(x) = L$, we have for every $\varepsilon > 0$, $|f(x) - L| < \varepsilon$, for some $\delta > 0$, $0 < |x - x_0| < \delta$. From reverse triangular inequality we get $||f(x)| - |L|| \leq |f(x) - L|$. Thus, $||f(x)| - |L|| < \varepsilon$, proving $\lim_{x \rightarrow x_0} |f(x)| = |L|$. $\square$

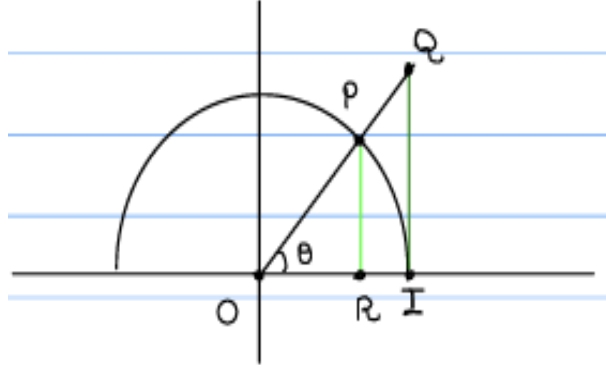
Theorem 7.1 (Squeeze/ Sandwich theorem). Let $X \subseteq \mathbb{R}$, $x_0 \in \mathbb{R}$ be a limit point of X , $f, g, h : X \rightarrow \mathbb{R}$ such that $f(x) \leq g(x) \leq h(x)$ for all $x \in X$. For suppose that $L = \lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} h(x)$. Then $\lim_{x \rightarrow x_0} g(x) = L$.

Proof. One just needs to use the squeeze theorem for sequences at real number. $\square$

In what follows we are going to introduce many fundamental limits, which appear in many applications and as a result of computing more complex limits.

Example 7.4. $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$.

We assume that all trigonometric functions exist and all results of trigonometry apply (later on we will derive them analytically)



We know that $\sin \theta$ is the signed length of the projection of OP into the y-axis, and that $\cos \theta$ is the signed length of the projection of OP into the x-axis. In what follows we fix $\theta \in (0, \frac{\pi}{2})$.

Consider the $\triangle OPR$, the sector $\widehat{OPI}$ and the triangle $\triangle OQI$. Supposing that we know what are of triangles, sectors we have are following

$$\text{area}(\triangle OPR) \leq \text{area}(\widehat{OPI}) \leq \text{area}(\triangle OQI)$$

But,

$$\text{area}(\triangle OPR) = \frac{\overline{OR} \cdot \overline{PR}}{2} = \frac{\sin \theta \cdot \cos \theta}{2}$$

$$\text{area}(\widehat{OPI}) = \frac{\theta}{2}$$

$$\text{area}(\triangle OQI) = \frac{\overline{OI} \cdot \overline{QI}}{2} = \frac{\tan \theta}{2}$$

Hence

$$\frac{\sin \theta \cos \theta}{2} \leq \frac{\theta}{2} \leq \frac{\tan \theta}{2}$$

whenever $\theta \in (0, \frac{\pi}{2})$.

So $\cos \theta \leq \frac{\theta}{\sin \theta} \leq \frac{1}{\cos \theta}$ for all $\theta \in (0, \frac{\pi}{2})$ or $\frac{1}{\cos \theta} \geq \frac{\sin \theta}{\theta} \geq \cos \theta$ for all $\theta \in (0, \frac{\pi}{2})$. We assume that $\lim_{\theta \rightarrow 0^+} \cos \theta$ exists. Since $\lim_{\theta \rightarrow 0^+} \frac{1}{\cos \theta} = 1$ by squeeze/ sandwich theorem we conclude that $\lim_{\theta \rightarrow 0^+} \frac{\sin \theta}{\theta} = 1$.

7.2. Continuity.

Definition 7.4. Let $X \subseteq \mathbb{R}$, $x_0 \in X$ be a limit point of X , and $f : X \rightarrow \mathbb{R}$ be a function. f is said to be continuous at x_0 if, and only if, the $\lim_{x \rightarrow x_0} f(x)$ exists and it is equal to $f(x_0)$.

In other words, for every $\varepsilon > 0$ there exists $\delta = \delta(x_0, \varepsilon) > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ for all $x \in X$ such that $|x - x_0| < \delta$.

If a function $f : X \rightarrow \mathbb{R}$ is continuous at each point of X (its domain) we say that f is continuous.

A function which is not continuous at a point x_0 is discontinuous at x_0 ; a function which is not continuous it is discontinuous.

Example 7.5. Any polynomial function $P(x) = a_0 + a_1x + \dots + a_nx^n$ for $x \in \mathbb{R}$ is continuous.

In fact, notice that for each $x_0 \in \mathbb{R}$ one fixes let's consider the following.

$$|P(x) - P(x_0)| = \left| \sum_{i=1}^n a_i (x^i - x_0^i) \right| = \left| \sum_{i=1}^n a_i (x - x_0) (x^{i-1} + x^{i-2}x_0 + \dots + x_0^{i-1}) \right|$$

$$\begin{aligned}
&= \left| \sum_{i=1}^n \left(a_i (x - x_0) \sum_{j=0}^{i-1} x^j x_0^{i-1-j} \right) \right| \leq \left(\sum_{i=1}^n |a_i| \sum_{j=0}^{i-1} |x|^j |x_0|^{i-1-j} \right) |x - x_0| \\
&= \psi(x) \cdot |x - x_0| \quad \text{for all } x \in \mathbb{R}.
\end{aligned}$$

Since we want to know what happens very close to x_0 , let's fix $|x - x_0| < 1$. Then $|x| < 1 + |x_0|$. Notice that restricting the domain of $\phi(x)$ to $(-1 - |x_0|, 1 + |x_0|)$ then $\phi(x)$ is bounded, that is, there exists some $M > 0$ such that $|\phi(x)| \leq M$ for all $x \in (-1 - |x_0|, 1 + |x_0|)$.

In fact, since $|x| < 1 + |x_0|$

$$\begin{aligned}
|\phi(x)| &= \sum_{i=1}^n |a_i| \sum_{j=0}^{i-1} |x|^{i-j} |x_0|^j = \sum_{i=1}^n \sum_{j=0}^{i-1} |a_i| \cdot |x|^{i-j} \cdot |x_0|^j \\
&< \sum_{i=1}^n \sum_{j=0}^{i-1} |a_i| (1 + |x_0|)^{i-j} \cdot |x_0|^j < \sum_{i=1}^n \sum_{j=0}^{i-1} (1 + |a_i|) (1 + |x_0|)^{i-j} (1 + |x_0|)^j
\end{aligned}$$

Then for all $x \in (-1 - |x_0|, 1 + |x_0|)$ we have that $|P(x) - P(x_0)| \leq M \cdot |x - x_0|$ for

$$M = \sum_{i=1}^n \sum_{j=0}^{i-1} (1 + |a_i|) (1 + |x_0|)^{i-j} (1 + |x_0|)^j \geq 1$$

Given any $\varepsilon > 0$ if we make $M |x - x_0| < \varepsilon$, we are guaranteeing that $|P(x) - P(x_0)| < \varepsilon$. To do so we require $|x - x_0| < \frac{\varepsilon}{M}$. Now we define $\delta := \min\{1, \frac{\varepsilon}{M}\}$. Therefore whenever $|x - x_0| < \delta$ it follows that $|P(x) - P(x_0)| < \varepsilon$. This proves that $\lim_{x \rightarrow x_0} f(x) = f(x_0)$, and then f is continuous at x_0 . Since x_0 was arbitrary we conclude that f is continuous for all $x_0 \in \mathbb{R}$, that is, f is continuous.

Example 7.6. Prove that the function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^2$ for all $x \in \mathbb{R}$ is continuous.

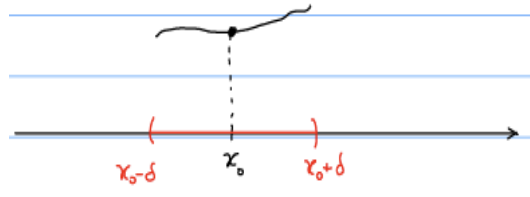
Fix any $x_0 \in \mathbb{R}$ and consider $|f(x) - f(x_0)| = |x^2 - x_0^2| = |x + x_0| \cdot |x - x_0| \leq (|x| + |x_0|) \cdot |x - x_0|$, which is true for all $x \in \mathbb{R}$. If we make the $|x - x_0| < 1$, then $|x| < 1 + |x_0|$. Therefore $|f(x) - f(x_0)| < (1 + |x_0| + |x_0|) \cdot |x - x_0| = (2|x_0| + 1) |x - x_0|$ for all $x \in \mathbb{R}$ such that $|x - x_0| < 1$.

Given any $\varepsilon > 0$ note that if we make $(2|x_0| + 1) \cdot |x - x_0| < \varepsilon$, we guarantee that $|f(x) - f(x_0)| < \varepsilon$. To do so we must have $|x - x_0| < 1$ and $|x - x_0| < \frac{\varepsilon}{1 + 2|x_0|}$, and if $\delta := \min\{1, \frac{\varepsilon}{1 + 2|x_0|}\}$, we have that whenever $|x - x_0| < \delta$ the $|f(x) - f(x_0)| < \varepsilon$. This proves that $\lim_{x \rightarrow x_0} f(x) = f(x_0)$ for all $x_0 \in \mathbb{R}$, then f is continuous.

Proposition 7.3. Let $X \subseteq \mathbb{R}, x_0 \in X$ be a limit point of X , $f, g : X \rightarrow \mathbb{R}$ be continuous functions at x_0 . Then:

- $f \pm g$ is continuous at x_0 ;
- $f \cdot g$ is continuous at x_0 ;
- If $g(x_0) \neq 0$ then $\frac{f}{g}$ is continuous at x_0 ;

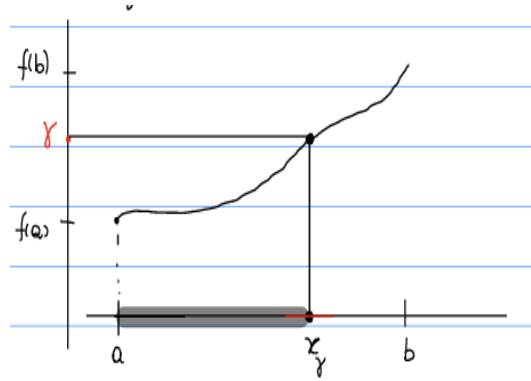
Proposition 7.4 (Local Preservation of Sign). Let $f : \mathbb{I} \rightarrow \mathbb{R}$ be continuous on the interval $\mathbb{I}$. If for some $x_0 \in \mathbb{I}$ we have $f(x_0) > 0$ there exists some $\delta > 0$ such that $f(x) > 0$ for all $x \in \mathbb{I} \cap (x_0 - \delta, x_0 + \delta)$



Proof. Since f is continuous given $\varepsilon = \frac{f(x_0)}{2} > 0$ there exists some $\delta = \delta\left(\frac{f(x_0)}{2}\right) > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ for all $x \in \mathbb{I} \cap (x_0 - \delta, x_0 + \delta)$. Then $f(x_0) - \varepsilon = \frac{f(x_0)}{2} < f(x) < \frac{3f(x_0)}{2}$ for all $x \in \mathbb{I} \cap (x_0 - \delta, x_0 + \delta)$. And since $f(x_0) > 0$ we are done. $\square$

In particular if $f(x_0) > \alpha$ for some $\alpha \in \mathbb{R}$ there exists $\delta > 0$ such that $f(x) > \alpha$ for all $x \in \mathbb{I} \cap (x_0 - \delta, x_0 + \delta)$. To prove it consider $l(x) := f(x) - \alpha$. Then $l(x_0) > 0$ by the local preservation of sign proposition above the result is immediate.

All continuous functions defined on an interval $[a, b]$ such that $f(a) < f(b)$ (or $f(b) < f(a)$) have the following property: · Given $\gamma \in (f(a), f(b))$ (or $\gamma \in (f(b), f(a))$) there exists $x_\gamma \in (a, b)$ such that $f(x_\gamma) = \gamma$



We are going to show that all continuous functions have the property above, and that is the intermediate value theorem.

For that we consider the following idea: if we look at the graph above and consider the set $\{x \in [a, b] : f(x) \leq \gamma\}$ the x_{gamma} is the supremum of this set and $f(x_\gamma) = \gamma$ as we can see in the picture above. Let's define $L := \{x \in [a, b] : f(x) \leq \gamma\}$. Note that $L \neq \emptyset$ because $a \in L$ since $f(a) < \gamma$, and since $L \subseteq [a, b]$ we have that L is bounded, in particular it is bounded above. By the completeness axiom there exists the least upper bound/ supremum of A , let's call it x_γ ($x_\gamma = \sup(L) = \text{glb}(L)$). Our claim is that $f(x_\gamma) = \gamma$. To prove it we show that the cases $f(x_\gamma) > \gamma$ and $f(x_\gamma) < \gamma$ don't happen.

· If $f(x_\gamma) > \gamma$ then there exists some $r > 0$ such that for every $x \in [a, b] \cap (x_\gamma - r, x_\gamma + r)$ we have $f(x) > \gamma$. But since $x_\gamma = \sup(L) = \text{glb}(L)$ for each $\varepsilon > 0$ there exists some $x_\varepsilon \in L$ such that $x_\gamma - \varepsilon < x_\varepsilon \leq x_\gamma$. And since $x_\varepsilon \in L$ we must have $f(x_\varepsilon) \leq \gamma$. In particular if we let $\varepsilon = r$ there must be $x_r \in L$ such that $x_\gamma - r < x_r \leq x_\gamma$ and $f(x_r) \leq \gamma$. Which contradicts that $f(x) > \gamma$ for all $x \in [a, b] \cap (x_\gamma - r, x_\gamma + r)$. Then $f(x_\gamma) > \gamma$ is false.

· If $f(x_\gamma) < \gamma$ there exists some $\bar{r} > 0$ such that for every $x \in [a, b] \cap (x_\gamma - \bar{r}, x_\gamma + \bar{r})$ we have $f(x) < \gamma$. In particular for any $x \in [a, b] \cap [x_\gamma, x_\gamma + \bar{r})$ we have $f(x) < \gamma$ and for that reason each $x \in [a, b] \cap [x_\gamma, x_\gamma + \bar{r})$ are also elements of L . Thus the assumption that $f(x_\gamma) < \gamma$ lead us to the conclusion that there are elements $x \in [a, b] \cap [x_\gamma, x_\gamma + \bar{r})$ such that $f(x) < \gamma$, which implies there $x \in L$. ($[a, b] \cap [x_\gamma, x_\gamma + \bar{r}) \subseteq L$). But since x_γ is the supremum/ g.l.b of L it is, in particular, an upper bound of L , which contradicts the previous that $[a, b] \cap [\overline{x_\gamma}, \overline{x_\gamma} + \bar{r}) \subseteq L$. Hence $f(x_\gamma) < \gamma$ is false.

The only left is that $f(x_\gamma) = \gamma$.

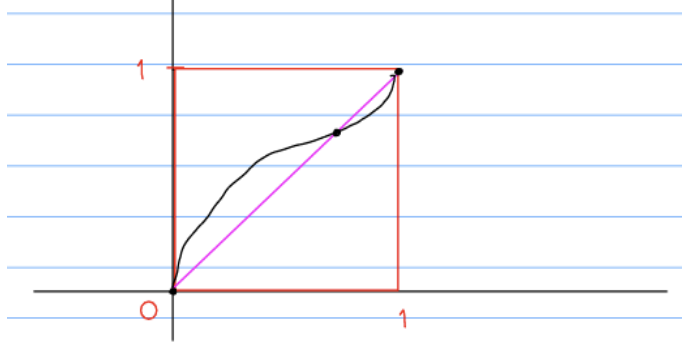
Example 7.7. Consider the polynomial function $\mathcal{P} : \mathbb{R} \rightarrow \mathbb{R}$ given by $\mathcal{P}(x) = x^7 - 20x^5 + 10$ for all $x \in \mathbb{R}$. Prove that $\mathcal{P}(x) = 0$ for some $x \in \mathbb{R}$.

Note that $\mathcal{P}(0) = 0 - 20 \cdot 0 + 10 = 10 > 0$ and $\mathcal{P}(1) = 1 - 20 + 10 = -9 < 0$. Since $\mathcal{P}$ is continuous, the restriction of $\mathcal{P}$ to $[0, 1]$, that is, the function $\mathcal{P}_{[0,1]} : \mathbb{R} \rightarrow \mathbb{R}$ given by $\mathcal{P}_{[0,1]} := x^7 - 20x^5 + 10$ for all $x \in [0, 1]$, is a continuous function such that $\mathcal{P}_{[0,1]}(0) > 0$ and $\mathcal{P}_{[0,1]}(1) < 0$.

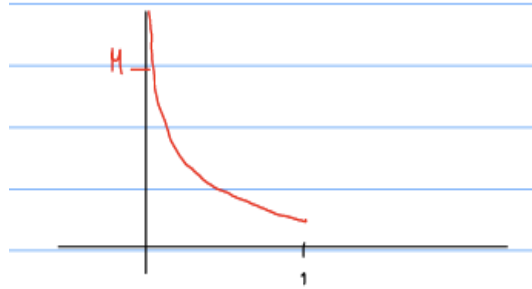
By the intermediate value theorem there exists some $x \in [0, 1]$ such that $\mathcal{P}_{[0,1]}(x) = \mathcal{P}(x) = x^7 - 20x^5 + 10 = 0$. This proves there exists a root for $\mathcal{P}(x)$ polynomial function.

Example 7.8. Let $f : [0, 1] \rightarrow [0, 1]$ be continuous. Prove that there exists some $x \in [0, 1]$ such that $f(x) = x$

Define $g : [0, 1] \rightarrow [0, 1]$ by $g(x) = f(x) - x$ for all $x \in [0, 1]$. Then $0 \leq g(0) = f(0) \leq 1$ and $-1 \leq g(1) = f(1) - 1 \leq 0$. g is the sum of two continuous functions on $[0, 1]$, therefore g is continuous. By the intermediate value theorem there exists some $x \in [0, 1]$ such that $g(x) = 0$



Theorem 7.2 (Extreme Value Theorem). Every continuous function $f : [a, b] \rightarrow \mathbb{R}$ attains its maximum and minimum values, that is, there exist $x_{\min} \in [a, b], x_{\max} \in [a, b]$ such that $f(x_{\min}) = \inf\{f(x) : x \in [a, b]\}, f(x_{\max}) = \sup\{f(x) : x \in [a, b]\}$. Note that the interval must be closed because the function $f : [0, 1] \rightarrow \mathbb{R}$ given by $f(x) = \frac{1}{x}$ for all $x \in [0, 1]$ is continuous but there is no $x_{\min} \in (0, 1]$ for which $f(x_{\min}) = \sup\{f(x) : x \in [a, b]\}$



Before we prove it, we need to auxiliary results:

Lemma 7.1. Every compact set in $(\mathbb{R}, |\cdot|)$ has a maximum and minimum elements, that is, if $K \subseteq \mathbb{R}$ is compact, there are $\underline{x} \in K, \bar{x} \in K$ such that for all $y \in K$ we have $\underline{x} \leq y \leq \bar{x}$.

Proof. Since K is compact in $(\mathbb{R}, |\cdot|)$, by Heine-Borel theorem, K is bounded and closed in $(\mathbb{R}, |\cdot|)$. K being bounded means that there exists $M > 0$ such that $K \subseteq [-M, M]$. Therefore by the Archimedean/ completeness axiom of $\mathbb{R}$ the $\sup(K) = \text{lub}(K)$ and $\inf(K) = \text{glb}(K)$ exist as real numbers, and moreover we know that $\inf(K) \leq y \leq \sup(K)$ for all $y \in K$.

We claim $\inf(K), \sup(K) \in K$. In fact, for each $n \in \mathbb{N}$ we have $\sup(K) - \frac{1}{n}$ is not an upper bound of K , then there must be some $y_n \in K$ such that $\sup(K) - \frac{1}{n} < y_n \leq \sup(K)$. Then $(y_n)_{n \in \mathbb{N}}$ is a sequence in K such that $|\sup(K) - y_n| \leq \frac{1}{n}$ for all $n \in \mathbb{N}$. Hence $\lim_{n \rightarrow \infty} y_n = \sup(K)$. By the sequential characterization of closed sets theorem in page 40 we have that $\sup(K) \in K$. And we put $\bar{x} = \sup(K)$.

Likewise, for the infimum we put $\underline{x} = \inf(K)$. This proves the result $\square$

Lemma 7.2. Let $f : A \longrightarrow \mathbb{R}$ be continuous, $A \subseteq \mathbb{R}$. If $K \subseteq A$ is a compact set, then $f(K)$ is also a compact set (both in $(\mathbb{R}, |\cdot|)$)

Proof. Notice that we have to prove that every sequence in $f(K)$ there exists a convergent subsequence of it. Then let $(y_n)_{n \in \mathbb{N}}$ be any sequence in $f(K)$ (that is $y_n \in f(K)$ for all $n \in \mathbb{N}$). Since $y_n \in f(K)$ there exists $x \in K$ such that $y_n = f(x)$, choose any of those x , fix it and denote it by x_n . Then $(x_n)_{n \in \mathbb{N}}$ is a sequence of points in K such that $y_n = f(x_n)$ for all $n \in \mathbb{N}$. Using that K is compact, and that $x_n \in K$ there exists a subsequence $(x_{n_j})_{j \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ which converges to some $l \in K$. If we consider the subsequence $(f(x_{n_j}))_{j \in \mathbb{N}} = (y_{n_j})_{j \in \mathbb{N}}$ of $(y_n)_{n \in \mathbb{N}}$, then by sequential continuity $\lim_{j \rightarrow \infty} y_{n_j} = \lim_{j \rightarrow \infty} f(x_{n_j}) = f(l)$. Thus we obtained a convergent subsequence of $(y_n)_{n \in \mathbb{N}}$. Hence $f(K)$ is compact in $(\mathbb{R}, |\cdot|)$ $\square$

Now we prove the extreme value theorem (EVT)

Proof. By lemma 2 above $f([a, b])$ is compact in $(\mathbb{R}, |\cdot|)$, and by lemma 1 above, there are $\underline{y}, \bar{y} \in f([a, b])$ such that $\bar{y} = \sup(f([a, b]))$, $\underline{y} = \inf(f([a, b]))$. And since $\underline{y}, \bar{y} \in f([a, b])$ there are $x_{\min} \in [a, b]$, $x_{\max} \in [a, b]$ such that $f(x_{\min}) = \underline{y}$ and $f(x_{\max}) = \bar{y}$. This proves the desired result. $\square$