

DIRECTED READING IN INTRODUCTORY COMBINATORICS

JUAN SILVERA, IMANA VELLATHOTTAM, SHAURYA PATIL

Protégés of AMMOC - An International Math Circle

CONTENTS

1. Set 1: Combinatorial Identities	3
2. Set 2: Combinatorial Identities 2	7
3. Set 3: Combinatorial Identities 3	10
4. Set 4: Multinomial Theorem	12
5. Set 5: Pigeonhole Principle	13
6. Set 6: Pigeonhole Principle 2	17
7. Set 7: Pigeonhole Principle 3	18
8. Set 8: Pigeonhole Principle 4	26
9. Set 9: Permutations and Combinations	35
10. Set 10: Permutations and Combinations	40
11. Acknowledgement	51

Conventions. We use $\mathbb{N} = \{1, 2, \dots\}$. Parameters counting objects or indexing sums are integers. For $a \geq 0$, we set $\binom{a}{b} = 0$ when $b < 0$ or $b > a$. A geometric partition assigns a point on a shared boundary to just one of its incident cells. In three-point area problems, collinear triples are allowed and have area zero. For counting nondegenerate triangles, collinear triples are excluded. The additional lemmas and the explicitly marked model or interpretation choices were added during revision.

1. SET 1: COMBINATORIAL IDENTITIES ¹

Problem 1.1 (1a). Derive Newton's Identity: $\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$, where n, k, r are integers and $0 \leq r \leq k \leq n$.

Solution 1.1 (1a). Question: How many ways are there to choose a subset of size r inside a subset of size k from a set of n distinguishable apples?

Left-hand side: First, choose k apples from the n total apples. There are $\binom{n}{k}$ ways to make this selection. Then select r apples from the k chosen apples. There are $\binom{k}{r}$ ways to do this. By the multiplication principle, the total number of ways to select a subset of size r inside a subset of size k is $\binom{n}{k}\binom{k}{r}$.

Right-hand side: Choose r apples first. There are $\binom{n}{r}$ ways to do this. Next, choose $k - r$ apples from the remaining $n - r$ apples to complete the k subset. This can be done in $\binom{n-r}{k-r}$ ways. By the multiplication principle, the total number of ways using this method is $\binom{n}{r}\binom{n-r}{k-r}$.

Since the left- and right-hand sides both count the total number of ways to select r apples inside a subset of size k apples from n , the combinatorial identity holds.

Problem 1.2 (1b). $\binom{n}{k} = \frac{n-k+1}{k}\binom{n}{k-1}$, where n, k are integers and $1 \leq k \leq n$.

Solution 1.2 (1b). Question: How many ways are there to choose a basket of size k apples from n apples?

Left-hand side: By definition of binomial coefficients, there are $\binom{n}{k}$ ways to do this.

Right-hand side: First, choose a basket of $k - 1$ apples from the n total apples. There are $\binom{n}{k-1}$ ways to do this. After this, there are $n - (k - 1) = n - k + 1$ apples remaining. Adding one of the remaining apples to the basket of size $k - 1$ forms a basket of size k . However, each final basket of size k can be formed in k different ways by choosing which apple was most recently added, so we must divide by k to avoid over-counting. By multiplication principle, the total number of baskets of size k is $\frac{n-k+1}{k}\binom{n}{k-1}$ ways.

Since the left- and right-hand sides both count the total number of ways to select baskets of size k apples from n apples, the combinatorial identity holds.

Problem 1.3 (2). If $2 \leq k \leq n - 2$, show that $\binom{n}{k} = \binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k}$, $n \geq 4$

Solution 1.3 (2). Question: How many ways are there to choose a basket of size k apples from n apples?

Left-Hand side: By definition of binomial coefficients, there are $\binom{n}{k}$ ways to do this.

Right-Hand side: Consider the set of n apples. All apples are distinguishable. All are red, except for two special apples: one green and one yellow.

Case 1: Both the green and yellow apples are in the basket.

Choose $k - 2$ apples from the remaining $n - 2$ apples. There are $\binom{n-2}{k-2}$ ways to do this.

¹Elementary Number Theory by David M. Burton Ch1

Case 2: Exactly one of the yellow and green apples is in the basket.

There are two choices for which special apple is included, then choose $k - 1$ apples from the remaining $n - 2$ apples. This results in $2\binom{n-2}{k-1}$ ways.

Case 3: Neither the yellow nor the green apple is in the basket. Choose k apples from the remaining $n - 2$ apples. There are $\binom{n-2}{k}$ ways to do this.

By addition principle, the three mutually exclusive cases are summed to total all baskets of size k .

Since the left- and right-hand sides both count the total number of ways to select baskets of size k apples from n apples, the combinatorial identity holds.

Problem 1.4 (3a). For an integer $n \geq 0$, prove that $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n$

Solution 1.4 (3a). We have a set A of n elements. Find the total number of subsets of Set A .

Method 1: Each element has 2 choices : It can be in the set or not. Thus, there are 2^n subsets of Set A .

Method 2: We can break down the number of subsets by how many elements they possess. If they possess k elements, the number of subsets is $\binom{n}{k}$. Thus, the total number of subsets is $\sum_{k=0}^n \binom{n}{k}$, including the empty subset when $k = 0$.

Thus, $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n$

Problem 1.5 (3b). For an integer $n \geq 1$, prove that $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \dots + (-1)^n \binom{n}{n} = 0$

Solution 1.5 (3b). This equation can be transformed into:

$$\sum_{\substack{0 \leq j \leq n \\ j \text{ even}}} \binom{n}{j} = \sum_{\substack{0 \leq j \leq n \\ j \text{ odd}}} \binom{n}{j}.$$

We have a set A with n elements. We make subsets of the first $n - 1$ elements of A . Now we call the last element x . For each such subset B , pair B with $B \cup \{x\}$. One of these two subsets has an even number of elements and the other has an odd number of elements, regardless of the size of B . Every subset of A belongs to exactly one such pair. Thus, the number of subsets with an even number of elements is equal to the number of subsets with an odd number of elements. In other words,

$$\sum_{\substack{0 \leq j \leq n \\ j \text{ even}}} \binom{n}{j} = \sum_{\substack{0 \leq j \leq n \\ j \text{ odd}}} \binom{n}{j}, \quad \sum_{j=0}^n (-1)^j \binom{n}{j} = 0.$$

Problem 1.6 (3c). For an integer $n \geq 1$, prove that $n2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \dots + n\binom{n}{n}$

Solution 1.6 (3c). We solve this identity by posing the following question: There are n sailors on an island with one ship. This ship need one captain and any number of other sailors on it. The captain is included when we count the total number of people on the ship. Find the total number of possible arrangements.

Left Hand Side: The first way to solve this is by first picking the captain and then the sailors. We have n ways of picking the captain. We then need to find the number of ways to pick the sailors. Each of the remaining $n - 1$ sailors have 2 choices: either be on the ship or not. Thus, there are 2^{n-1} ways to choose the sailors and $n2^{n-1}$ total arrangements.

Right Hand Side: Another way we could approach this problem is by breaking it down by total number of people on ship and then choosing the captain. Obviously, for k people, there are $k\binom{n}{k}$ ways so summing for all values of k , we get that there are $\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \dots + n\binom{n}{n}$ ways.

Thus, $\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \dots + n\binom{n}{n} = n2^{n-1}$

Problem 1.7 (3d). For an integer $n \geq 0$, prove that $\binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + \dots + 2^n\binom{n}{n} = 3^n$

Solution 1.7 (3d). Referring back to [Solution 1.6](#), we can solve this problem by posing a similar question: There are n sailors on an island with one ship. Choose a group of r sailors from n , and then from that group each sailor may become a captain or not. Find the total number of possible arrangements.

Left-hand side: For a group of sailors of size r , there are $\binom{n}{r}$ ways to choose the sailors. Since there can be up to as many captains as there are sailors, each sailor independently has the choice to either become a captain or not. This gives $2^r\binom{n}{r}$ possible arrangements. Thus, summed for all values of r from 0 to $r = n$, the left-hand side equals $\binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + \dots + 2^n\binom{n}{n}$.

Right-hand side: Each islander has three choices: stay on the island, be a sailor, or be a captain. Total number of possible arrangements that can be formed is $= 3^n$.

Since the left- and right-hand sides both count the number of ways islanders can be arranged on this ship, the combinatorial identity holds.

Problem 1.8 (3e). For an integer $n \geq 1$, prove that $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \binom{n}{6} + \dots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots = 2^{n-1}$

Solution 1.8 (3e). By [Solution 1.5](#), $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \binom{n}{6} + \dots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots$ By [Solution 1.4](#), $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n$. As established, the sum of binomial coefficients with even lower index is equal to the sum with odd lower index, and each are half of the total number of subsets of n . Since $\frac{2^n}{2} = 2^{n-1}$, the combinatorial identity holds.

Problem 1.9 (3f). For an integer $n \geq 0$, prove that $\sum_{r=0}^n \frac{(-1)^r}{r+1} \binom{n}{r} = \frac{1}{n+1}$.

Solution 1.9 (3f). Let there be 1 player, Crazy Dave, gambling against n other players. We choose a ranking of all $n+1$ players uniformly at random, with no ties, and the first player wins. Each player is equally likely to win, so Dave has a $\frac{1}{n+1}$ chance of winning.

Now, let E_i mean that opponent i is before Dave in the ranking. For any chosen r opponents, all of them are before Dave exactly when Dave is last among these $r+1$ players. That has probability $\frac{1}{r+1}$. There are $\binom{n}{r}$ ways to choose those opponents.

We start with probability 1 and use inclusion-exclusion to remove the events in which at least one opponent is before Dave. The probability that none is before him is therefore

$$\sum_{r=0}^n \frac{(-1)^r}{r+1} \binom{n}{r}.$$

This is also the probability that Dave wins. Hence, the equality must hold. The same calculation gives 1 on both sides when $n = 0$.

2. SET 2: COMBINATORIAL IDENTITIES 2 ²

Problem 2.1 (24). For an integer $n \geq 0$, prove that $\sum_{r=0}^n 3^r \binom{n}{r} = 4^n$

Solution 2.1 (24). We can think of this situation as there are n elements in Set A. Now, Set A has a subset called Set B. Set B has another Subset called Set C and Set C has another subset called Set D. Find the total number of combinations of elements in each Set.

The first way of approaching this problem is by giving each element a_i in Set A four options: It is either in all 4 sets, in the first 3 sets, in Sets A and B or only in Set A. Thus, the total number of combinations of elements is 4^n .

The second way of approaching this problem is by breaking down the number of elements in Set B and so on to get the total number of combinations. We know that if Set B has r elements provided $0 \leq r \leq n$, then there are $\binom{n}{r}$ ways of forming Set B. Now, each element in Set B has three options: either be in all 4 sets, be in the first three sets (A,B and C), or just be in Sets A and B. Thus, the total number of ways to form Sets B, C and D given that Set B has r elements is $\binom{n}{r} 3^r$. The total number of ways is then $\sum_{r=0}^n 3^r \binom{n}{r}$. The equality is now proved.

Problem 2.2 (25). For an integer $n \geq 0$, prove that $\sum_{r=0}^n (r+1) \binom{n}{r} = (n+2)2^{n-1}$

Solution 2.2 (25). For $n = 0$, both sides equal 1. Now suppose $n \geq 1$. Referring back to [Solution 1.6](#), we can solve this problem by employing a similar setup: There are n sailors on an island with one ship. Choose a group of r sailors from n , and then select one captain from that group or optionally none. Find the total number of possible arrangements.

Left-hand side: For a group of sailors of size r , there are $\binom{n}{r}$ ways to choose the sailors. Since there is the option to have no captain, there are $r+1$ ways to choose the captain. This gives $(r+1)\binom{n}{r}$ possible arrangements, summed for all values of r from 0 to $r = n$.

Right-hand side: Alternatively, if there is no captain, there are 2^n ways to choose sailors. If there is a captain, there are n ways to choose the captain and 2^{n-1} ways to choose the sailors on the ship. By addition principle, there are $2^n + n2^{n-1} = (n+2)2^{n-1}$ possible arrangements with this approach.

Since the left- and right-hand sides both count the number of arrangements of islanders on the ship, the combinatorial identity holds.

Problem 2.3 (26). For an integer $n \geq 0$, prove that $\sum_{r=0}^n \frac{1}{r+1} \binom{n}{r} = \frac{1}{n+1} (2^{n+1} - 1)$

Solution 2.3 (26). We pose the problem that Jack is sitting with n gamblers. For each nonempty subset of these $n+1$ people, we hold one game with exactly that subset playing. In each game, we choose the winner uniformly from the players in that game. Let W be the total number of games Jack wins. Find the expected value of W .

The first way to think about this problem is by adding Jack's expected wins over all the games. A game with Jack and r other players gives him a $\frac{1}{r+1}$ chance of winning, and there are $\binom{n}{r}$ such games. Games without Jack contribute zero. Thus,

$$E(W) = \sum_{r=0}^n \frac{1}{r+1} \binom{n}{r}.$$

²Principles and Techniques in Combinatorics by Chen Chuan-Chong, et. al Ch2

The second way to think about this problem is that all $n + 1$ people have the same expected number of wins. There are $2^{n+1} - 1$ nonempty subsets, and every game contributes exactly one win to the total. Thus, each person's expected number of wins is $\frac{1}{n+1}(2^{n+1} - 1)$. These are two calculations of the same $E(W)$ and the proof is complete.

Problem 2.4 (28). For integers $0 \leq m \leq n$, prove that $\sum_{r=m}^n \binom{n}{r} \binom{r}{m} = 2^{n-m} \binom{n}{m}$

Solution 2.4 (28). To prove this, we need to pose a question involving three different Sets : A , B and C . Set A has n elements, while Set B is a subset of Set A . Set C is a subset of Set B with m elements. We need to find the total number of combinations of elements in Sets A , B and C .

We can approach this problem by realizing that the number of elements in Set B is unknown. Thus, we assign the variable r to the number of elements in Set B , provided that $m \leq r \leq n$. Now, for any r , the total number of combinations is $\binom{n}{r} \binom{r}{m}$ to choose the elements in Set B and Set C . Thus, summing it over all possible values of r , we get the total number of combinations to be $\sum_{r=m}^n \binom{n}{r} \binom{r}{m}$

Another way to approach this problem is by choosing Set C from Set A . We can do this in $\binom{n}{m}$ ways. Now, Set B has all the elements in Set C and could also include the remaining elements in Set A . Each of the $n - m$ elements not in Set C can either be in Set B or not be in Set B . Thus, there is a total of 2^{n-m} ways to form Set B . Hence, the total number of ways to choose Sets B and C is $2^{n-m} \binom{n}{m}$ and the proof is complete.

Problem 2.5 (32). For an integer $n \geq 1$, prove that $\sum_{r=0}^{n-1} \binom{2n-1}{r} = 2^{2n-2}$

Solution 2.5 (32). We can see that $\binom{2n-1}{i} = \binom{2n-1}{2n-1-i}$ because the former is talking about the total number of ways we can choose i elements from $2n - 1$ elements and the latter is talking about the total number of ways we can exclude i elements from $2n - 1$ elements. Also, we can note that in $\sum_{r=0}^{n-1} \binom{2n-1}{r}$, each individual $\binom{2n-1}{i}$ in the summation has a counterpart $\binom{2n-1}{2n-1-i}$ not included in the summation and vice versa. Thus, $\sum_{r=0}^{n-1} \binom{2n-1}{r} = \frac{1}{2} \sum_{r=0}^{2n-1} \binom{2n-1}{r}$. Also, from [Solution 1.4](#), we know that $\frac{1}{2} \sum_{r=0}^{2n-1} \binom{2n-1}{r} = \frac{1}{2} 2^{2n-1} = 2^{2n-2}$. Hence, we have proved this identity.

Problem 2.6 (33). For an integer $n \geq 0$, prove that $\sum_{r=0}^n \binom{2n}{r} = 2^{2n-1} + \frac{1}{2} \binom{2n}{n}$

Solution 2.6 (33). For $n = 0$, both sides equal 1. Now suppose $n \geq 1$. By the symmetry $\binom{2n}{r} = \binom{2n}{2n-r}$, we know that $\sum_{r=0}^{n-1} \binom{2n}{r} = \sum_{r=n+1}^{2n} \binom{2n}{r}$. Also, $\sum_{r=0}^{2n} \binom{2n}{r} = \sum_{r=0}^{n-1} \binom{2n}{r} + \binom{2n}{n} + \sum_{r=n+1}^{2n} \binom{2n}{r}$. Upon substituting, we get that $\sum_{r=0}^{2n} \binom{2n}{r} = 2 \sum_{r=0}^{n-1} \binom{2n}{r} + \binom{2n}{n}$. We also know that $\sum_{r=0}^{2n} \binom{2n}{r} = 2^{2n}$ from [Solution 1.4](#). Thus, $2^{2n} = 2 \sum_{r=0}^{n-1} \binom{2n}{r} + \binom{2n}{n}$ and $2^{2n-1} = \sum_{r=0}^{n-1} \binom{2n}{r} + \frac{1}{2} \binom{2n}{n}$. By adding $\frac{1}{2} \binom{2n}{n}$ to both sides, we get that $\sum_{r=0}^n \binom{2n}{r} = 2^{2n-1} + \frac{1}{2} \binom{2n}{n}$ and the proof is complete.

Problem 2.7 (34). For an integer $n \geq 1$, prove that $\sum_{r=0}^n r \binom{2n}{r} = n 2^{2n-1}$

Solution 2.7 (34). By [Solution 1.6](#), we know that $\sum_{r=0}^n r \binom{2n}{r} = n 2^{2n-1}$. Hence, we propose a similar problem construction: There are $2n$ sailors on an island with one ship. Choose a group of r sailors from $2n$, and one captain from the selected sailors. Let r be summed from $r = 0$ to $r = n$. Find the total number of possible arrangements of islanders on the ship.

Left-hand side: For a group of sailors of size r , there are $\binom{2n}{r}$ ways to choose the sailors. From the selected sailors, there are r ways of choosing the captain. This gives $r \binom{2n}{r}$ possible arrangements, summed for all values of r from 0 to $r = n$.

Right-hand side: There are $2n$ ways of picking the captain. Since the captain is already on the ship, we may choose at most $n - 1$ of the remaining $2n - 1$ sailors. We pair each subset of those sailors with its complement. Exactly one subset in each pair has at most $n - 1$ members, because the two sizes sum to $2n - 1$. Thus, exactly half of the 2^{2n-1} subsets are allowed. This gives 2^{2n-2} choices after the captain is fixed, and $2n \cdot 2^{2n-2} = n2^{2n-1}$ arrangements in total. Since the left- and right-hand sides both count the number of arrangements of $2n$ islanders with a maximum of $r = n$ sailors on a ship, the combinatorial identity holds.

Problem 2.8 (36). For an integer $n \geq 1$, prove that $\sum_{r=1}^n \binom{n}{r} \binom{n-1}{r-1} = \binom{2n-1}{n-1}$

Solution 2.8 (36). There are n people in a room who can play 2 board games with the same number of players in both games. Jack is already in the second game and players can compete in both games. Find the total number of combinations of people playing both the games.

We can approach this problem by breaking it down case by case for values of r , which represents players in each game. Obviously, $r \geq 1$ because Jack is already in the second game. Now, for a particular r , we have $\binom{n}{r} \binom{n-1}{r-1}$ combinations, and in total we have $\sum_{r=1}^n \binom{n}{r} \binom{n-1}{r-1}$ combinations.

Another way to approach this problem is by making two lists of labeled copies of the people. The first list has all n people and represents people excluded from Game 1. The second list has all people except Jack and represents people included in Game 2 besides Jack. Copies in different lists are different objects, even when they stand for the same person.

If each game has r players, we select $n - r$ copies from the first list and $r - 1$ copies from the second list. Thus, we select $n - 1$ copies from $2n - 1$ copies in total.

Conversely, any selection of $n - 1$ copies tells us whom to exclude from Game 1 and whom to include in Game 2 together with Jack. If a copies came from the first list, both games then have $n - a$ players. Hence, every selection gives exactly one allowed pair of games, and the total is $\binom{2n-1}{n-1}$. The proof is complete.

3. SET 3: COMBINATORIAL IDENTITIES 3 ³⁴⁵

Problem 3.1 (Wichita 5). Let n, k be integers with $2 \leq k \leq n$. Give combinatorial proof of the identity $\binom{n}{2}\binom{n-2}{k-2} = \binom{n}{k}\binom{k}{2}$

Solution 3.1 (Wichita 5). We can solve this problem by proposing the following question: In how many ways are there to select a subcommittee of k people from n people, with two directors having the same status?

Left-hand side: First, select the two directors from the n total people. Then, choose the $k - 2$ more people required to form a subcommittee of size k from the remaining $n - 2$ people. This results in $\binom{n}{2}\binom{n-2}{k-2}$ possible arrangements.

Right-hand side: Another approach is to begin by selecting the k people from n that form the subcommittee. Next, select the two directors from the subcommittee k . Through this method, there are $\binom{n}{k}\binom{k}{2}$ possible arrangements.

Since the left- and right-hand sides both count the number of ways to form a subcommittee of size k from n people with two directors, the combinatorial identity holds.

Problem 3.2 (Wichita 9). Let $n \geq 2$ be an integer. Establish the identity below using a combinatorial proof.

$$(1) \quad \binom{2}{2}\binom{n}{2} + \binom{3}{2}\binom{n-1}{2} + \binom{4}{2}\binom{n-2}{2} + \dots + \binom{n}{2}\binom{2}{2} = \binom{n+3}{5}$$

Solution 3.2 (Wichita 9). We have the set $\{1, 2, \dots, n+3\}$ and want to pick 5 numbers out of it. Find all possible combinations. The first way to solve it is using basic combinatorics to get $\binom{n+3}{5}$ combinations. The second way to solve it is by picking the middle number. Obviously, it lies between 3 and $n+1$. If the middle number is i , then there are $\binom{i-1}{2}\binom{n-i+3}{2}$. Hence, there is $\binom{2}{2}\binom{n}{2} + \binom{3}{2}\binom{n-1}{2} + \binom{4}{2}\binom{n-2}{2} + \dots + \binom{n}{2}\binom{2}{2}$ ways to choose and the proof is complete.

Problem 3.3 (AwesomeMath Summer Program, Ch. 5, Theorem 5). Prove the Hockey Stick Theorem. If $n \geq k \geq 0$

$$(2) \quad \sum_{j=k}^n \binom{j}{k} = \binom{n+1}{k+1}$$

Solution 3.3. We propose the following question: How many ways are there to pick $k+1$ people to form a subcommittee from $n+1$ student leaders, each representing a different numbered school from 1 to $n+1$.

Left-hand side: Let the largest numbered school represented be $x+1$. The highest numbered school cannot be less than $k+1$ as there wouldn't be enough schools to choose from for the remaining k members of the subcommittee. Therefore $x+1 = k+1, k+2, \dots, n+1$, and fixing $x+1$ gives $\binom{x}{k}$ ways to choose the remaining k members. Then, summing over all values of x from k to n , we get the left-hand side.

³Discrete Math For Shockers by John Hammond

⁴AwesomeMath Summer Program

⁵Problems and Solutions in Mathematical Olympiad — High School 3, Chapter 2 by Hong-Bing Yu

Right-hand side: Count the number of ways to choose a $k + 1$ person subcommittee from $n + 1$ student representatives, which yields $\binom{n+1}{k+1}$ ways.

Since the left- and right-hand sides both count the number of ways to form a subcommittee of size $k + 1$ from $n + 1$ people, the combinatorial identity holds.

Problem 3.4 (High School 3, Ch. 2, No. 10). For an integer $n \geq 1$, Prove $\binom{2n}{2} = 2\binom{n}{2} + n^2$

Solution 3.4 (High School 3, Ch. 2, No. 10). We pose the problem of how many ways are there to choose 2 objects from a list of $2n$ objects. Obviously, we can do this in $\binom{2n}{2}$ ways. However, we can also break the $2n$ object list into two different n object lists. Now, we can either choose two objects from the same list or different list. The first way yields $2\binom{n}{2}$ ways where the factor 2 counts which list we're choosing from. The second way yields n^2 ways : n options from each set. Thus, $\binom{2n}{2} = 2\binom{n}{2} + n^2$

4. SET 4: MULTINOMIAL THEOREM ⁶

Problem 4.1 (Quick Check 1). What is the coefficient of $x_1x_2x_3$ in $(x_1 + 2x_2 + 3x_3)^3$?

Solution 4.1 (Quick Check 1). By the multinomial theorem, the term containing $x_1x_2x_3$ is obtained by choosing x_1 , $2x_2$ and $3x_3$ from the three factors, one from each factor. There are $3!$ ways to assign these choices to the factors. This gives a coefficient of

$$\frac{3!}{1!1!1!} \cdot 1 \cdot 2 \cdot 3 = 36.$$

⁶A Walk Through Combinatorics by Miklos Bona

5. SET 5: PIGEONHOLE PRINCIPLE ⁷

Problem 5.1 (Example 1). In a bag, there are some balls of the same size that are colored by 7 colors, and for each color the number of balls is 77. At least how many balls are needed to be picked out at random to ensure that one can obtain 7 groups of 7 balls each such that in each group the balls are homochromatic.

The seven groups must be disjoint, but different groups may have the same color.

Solution 5.1 (Example 1). Approach this problem by looking at how one homochromatic group is made: this happens when we pick 7 balls of one color. If we pick 13 balls of each of six colors and 6 balls of the seventh color, we have $6 \cdot 13 + 6 = 84$ balls but only six disjoint homochromatic groups. Thus, 84 balls do not always suffice.

Now let a_i be the number of balls picked of color i , and write $a_i = 7q_i + r_i$, where $0 \leq r_i \leq 6$. We can make q_i disjoint groups of that color. If we cannot make seven groups in total, then $\sum_{i=1}^7 q_i \leq 6$. Therefore,

$$\sum_{i=1}^7 a_i = 7 \sum_{i=1}^7 q_i + \sum_{i=1}^7 r_i \leq 7 \cdot 6 + 7 \cdot 6 = 84.$$

Thus, any selection of 85 balls must contain seven groups. The minimum number is **85**.

Problem 5.2 (Example 2). A bag contains 200 marbles. There are 60 red ones, 60 blue ones, 60 green, and the remaining 20 consist of yellow and white ones. If marbles are chosen from the bag without looking, what is the smallest number of marbles one must pick in order to ensure that, among the chosen marbles, at least 20 are of the same color?

Solution 5.2 (Example 2). Let there be y yellow marbles and w white marbles, where $y+w = 20$. We can apply the Pigeonhole principle by making one hole for each color. To avoid choosing 20 of one color, we can choose at most 19 red, 19 blue and 19 green marbles. We can also choose at most $\min(y, 19)$ yellow and $\min(w, 19)$ white marbles. This gives

$$57 + \min(y, 19) + \min(w, 19)$$

marbles without 20 of any color, and this number can be attained. One more marble forces the condition.

If both yellow and white occur, then $1 \leq y, w \leq 19$, so the answer is $57 + 20 + 1 = 78$. If all 20 of the remaining marbles are one color, the answer is $57 + 19 + 1 = 77$. Therefore, with the split unspecified, 78 is the smallest number that guarantees the condition for every allowed split.

Problem 5.3 (Example 3). If 51 numbers are arbitrarily taken out from the first 100 natural numbers $\{1, 2, \dots, 100\}$, prove that there must be two numbers drawn such that one is a multiple of the other.

Solution 5.3 (Example 3). Every number from $\{1, 2, \dots, 100\}$ is either divisible by 2 or not. By removing all factors of 2, we can represent each number uniquely in the form of $a = 2^m \cdot q$, where $m \geq 0$ is an integer and q is odd. For example, $1 = 2^0 \cdot 1$, $2 = 2^1 \cdot 1$, $3 = 2^0 \cdot 3$, $4 = 2^2$, $5 = 2^0 \cdot 5$, $6 = 2^1 \cdot 3$, $7 = 2^0 \cdot 7$... Notice that every odd number q is represented as $2^0 q$. Hence, each value of q for odd numbers corresponds to the 50 distinct odd numbers from the set. This sets up 50 pigeonholes. Now, when you select the 51st number, it is forced that two numbers share the same value of q . Say the two numbers are $2^e q$ and $2^f q$ with $e \leq f$. The larger divided by the smaller is 2^{f-e} , an integer, so one is a multiple of the other.

⁷Problems and standard solution methods adapted from Lecture Notes on Mathematical Olympiad Courses For Junior Section Vol. 2 by Xu Jiagu

Problem 5.4 (Example 8). If five points are taken at random from within a square of side 1, prove that there must be two of them such that the distance between them is not greater than $\frac{\sqrt{2}}{2}$.

Solution 5.4 (Example 8). Divide the square of side length 1 into four smaller sized congruent squares with side length $\frac{1}{2}$ by connecting the midpoints of the opposite sides. Consider each of the four squares as pigeonholes. There are five given points (pigeons) within the square of side 1. Therefore, one square of side length $\frac{1}{2}$ must hold two points. The maximum distance between two points in a square of side length a is its diagonal, calculated by $a\sqrt{2}$. Thus, the distance between the two points is not greater than $\frac{\sqrt{2}}{2}$, which completes the proof.

Problem 5.5 (Part A Question 1). Let n be a positive integer. Prove that among any $n + 1$ integers, there must be at least two which are congruent modulo n .

Solution 5.5 (Part A Question 1). Let the $n + 1$ integers be the pigeons. When an integer is divided by n , the possible remainders are $0, 1, 2, \dots, n - 1$, which are the pigeonholes. This gives n possible integer remainders. By Pigeonhole principle, two numbers from $n + 1$ must have the same remainder, when divided by n . Thus, there must be at least two which are congruent modulo n .

Problem 5.6 (Part A Question 2). Let $1 \leq a_1 \leq a_2 \leq a_3 \leq \dots \leq a_{n+1} \leq 2n$ be $n + 1$ integers, where $n \geq 1$. Prove that there must be two of them, $a_i \leq a_j$ with $i < j$ such that $a_i | a_j$.

Solution 5.6 (Part A Question 2). There are $n + 1$ integers that can be written uniquely in the form of $a = 2^m \cdot q$, where $m \geq 0$ and q is odd, as in [Solution 5.3](#). The possible values of q are $1, 3, \dots, 2n - 1$. Their number is

$$\frac{(2n - 1) - 1}{2} + 1 = n.$$

By pigeonhole principle, two of the $n + 1$ entries share the same odd part. Since the entries are in nondecreasing order, we can choose their indices as $i < j$ and write $a_i = 2^e q$, $a_j = 2^f q$ with $e \leq f$. Thus, $a_j / a_i = 2^{f-e}$ is an integer and $a_i | a_j$.

Problem 5.7 (Part A Question 3). Prove that in the 2009 numbers $1, 11, \dots, \underbrace{111\dots111}_{2009 \text{ times}}$, there is one that is divisible by 2009.

Solution 5.7 (Part A Question 3). We can solve this problem by contradiction. Let R_t be the number with t digits all equal to 1. If none of $R_1, R_2, \dots, R_{2009}$ is divisible by 2009, then there are only 2008 possible nonzero remainders. As there are 2009 numbers, by Pigeonhole principle, two numbers R_p and R_q , with $p < q$, have the same remainder. Their difference is divisible by 2009, and

$$R_q - R_p = 10^p R_{q-p}.$$

Since the GCF of 10^p and 2009 is 1, we can cancel 10^p in this divisibility statement. Therefore, 2009 divides R_{q-p} . But $1 \leq q - p \leq 2008$, so this is a number in our original list. This completes the contradiction.

Problem 5.8 (Part A Question 4). 19 points are taken randomly inside an equilateral triangle of area 1 m^2 . Prove that among the triangles formed by these points, there must be one with area not greater than $\frac{1}{9} \text{ m}^2$.

Solution 5.8 (Part A Question 4). Divide the triangle of area 1 m^2 into nine smaller congruent triangles of area $\frac{1}{9} \text{ m}^2$. Thus, we have 19 pigeons and 9 pigeonholes. By Pigeonhole principle, at least one pigeonhole contains at least $\lceil 19/9 \rceil = 3$ pigeons. It is impossible that the area formed

by these three points is larger than the area of the triangle in which it is contained. Therefore, there must be one triangle formed by the points with an area not greater than $\frac{1}{9} \text{ m}^2$.

Problem 5.9 (Part A Question 5). Prove that among any seven distinct integers, there must be two such that their sum or difference is divisible by 10.

Solution 5.9 (Part A Question 5). When dividing a number by 10, there are 10 possible remainders, ranging from 0 to 9. Now, set up the pigeonholes as such: (1) remainder of 0, (2) remainder of 1 or 9, (3) remainder of 2 or 8, (4) remainder of 3 or 7, (5) remainder of 4 or 6, and (6) remainder of 5. There are only six pigeonholes and seven integers, which means that one hole must have two numbers. With two numbers in the same hole, the proof is complete. If the two numbers have the same remainder, their difference is divisible by ten. If the two numbers have remainders that complement to ten, their sum is divisible by ten.

Problem 5.10 (Part A Question 8). Let $n \geq 1$ be an integer. When inserting $n+1$ points on the segment OA at random, prove that there must be two of them with a distance not greater than $\frac{1}{n}$ given that OA is 1.

Solution 5.10 (Part A Question 8). We arrange the $n+1$ chosen points along the segment in nondecreasing order. Let their coordinates be $0 \leq x_0 \leq x_1 \leq \dots \leq x_n \leq 1$. We have n gaps between consecutive chosen points, and their total length is

$$(x_1 - x_0) + (x_2 - x_1) + \dots + (x_n - x_{n-1}) = x_n - x_0 \leq 1.$$

Thus, the average gap is at most $\frac{1}{n}$, so at least one gap is at most $\frac{1}{n}$. The endpoints of that gap are two of the chosen points, which completes the proof.

Problem 5.11 (Part B Question 1). Prove that any subset of 20 numbers from the sequence $1, 4, 7, \dots, 100$ must contain two numbers that sum to 104.

Solution 5.11 (Part B Question 1). This is an arithmetic sequence with the general term $a_n = 1 + 3(n-1)$. Solving $100 = 1 + 3(n-1)$ gives $n = 34$ numbers in the sequence. Let the holes be the 16 pairs

$$\{4, 100\}, \{7, 97\}, \dots, \{49, 55\},$$

together with the two individual holes $\{1\}$ and $\{52\}$. These are 18 disjoint holes containing every term of the sequence. Therefore, among 20 selected numbers, two must be in the same hole. A hole with two different numbers is one of the pairs summing to 104, so the proof is complete.

In fact, 19 selected numbers already suffice. Choosing one number from each of the 18 holes shows that 18 need not suffice.

Problem 5.12 (Part B Question 2). On the blackboard some student has written 17 natural numbers, and their units digit is inside the set $\{0, 1, 2, 3, 4\}$. Prove that one can always select out 5 numbers such that their sum is divisible by 5.

Solution 5.12 (Part B Question 2). We can use the pigeonhole principle to solve this question. We approach this question by trying to prove the opposite and contradicting. Each of the 17 numbers is either 0, 1, 2, 3 or 4 mod 5. If we have 5 numbers which are all of the same residue modulo 5, then we can just choose those 5 numbers since the resulting residue will be $5x \text{ mod } 5$ which is 0. Thus, at most, we can only have 4 numbers for each residue class. Also, if we choose numbers from all residue classes, we can randomly pick one number from each class and the resulting sum will be $0 + 1 + 2 + 3 + 4 \equiv 0 \text{ mod } 5$. Thus, we can only have 4 residue classes chosen from at most, and 4 numbers from each class chosen from at most if we want to show that there is a possibility where the sum of 5 numbers is not divisible by 5. This is a total of $4 \cdot 4 = 16$ numbers at most. Thus, we cannot have 17 natural numbers, whose units digits is

inside the set $\{0, 1, 2, 3, 4\}$ and we are unable to pick 5 suitably chosen entries such that their sum is divisible by 5. Hence, the proof is complete.

Problem 5.13 (Part B Question 3). Prove that from a set of ten distinct two-digit numbers (in the decimal system), it is possible to select two disjoint subsets whose members have the same sum.

Solution 5.13 (Part B Question 3). The total number of subsets that can be formed from ten distinct two-digit numbers is $2^{10} = 1024$. The minimum sum of a nonempty proper subset is 10. The sum of a proper subset is at most $91 + 92 + 93 + 94 + 95 + 96 + 97 + 98 + 99 = 855$. This gives $855 - 10 + 1 =$ at most 846 possible sums of nonempty proper subsets. By Pigeonhole principle, there are 1022 nonempty proper subsets and 846 possible sums for those subsets. Thus, there are two subsets that share the same sum.

We must now prove that two disjoint subsets share the same sum. From any two subsets with an equal sum of elements, we can remove the elements they have in common by subtracting the intersection of elements from each set. This results in two distinct subsets. As the sum of the intersection in each set is the same, the two sets will still have an equal sum of elements. Furthermore, neither remaining subset can be empty. If one were empty, equality of sums and positivity of the elements would force the other to be empty too, making the original subsets equal. Therefore, there are two disjoint subsets whose members have the same sum.

6. SET 6: PIGEONHOLE PRINCIPLE 2 ⁸

Problem 6.1 (Cornell Problem 1). Suppose $n \geq 2$ baseball teams are playing in a tournament. If no two teams play each other more than once, prove that two teams have to play the same number of games.

Solution 6.1 (Cornell Problem 1). We use proof by contradiction. We can notice that the most number of games a team plays is $n-1$ games. This is if they play every other team once. Thus, each team could have played anywhere from 0 to $n-1$ games, which is n possible games played. We also notice that there are n teams, which means that the only way for no team to play the same number of games is if each value from 0 to $n-1$ is occupied by a team and vice versa. This also means that one team has played $n-1$ games and another has played 0. This is a contradiction because a team that plays $n-1$ games must have played every other team but in this case, another team has played 0 games. Thus, two teams have to have played the same number of games.

Problem 6.2 (Cornell Problem 2). Prove that there exist two different powers of 7 whose difference is divisible by 2003.

Solution 6.2 (Cornell Problem 2). We can prove this by selecting the 2004 distinct powers $7^0, 7^1, \dots, 7^{2003}$. Now, each power of 7 has one of the residues $0, 1, 2, \dots, 2002$ modulo 2003. This is a total of 2003 possible residues upon division by 2003. We also know that if two powers of 7 are the same mod 2003, then their difference is divisible by 2003. Thus, by the Pigeonhole Principle, we can prove that two different powers of 7 have a difference divisible by 2003.

Problem 6.3 (Cornell Problem 3). Fifty-one points are placed in a square of side length 1. Prove that there is a disk of radius $\frac{1}{7}$ that contains at least three of those points.

Solution 6.3 (Cornell Problem 3). We divide the square into 25 smaller congruent squares of side length $\frac{1}{5}$. There are 51 points, so by the pigeonhole principle, one smaller square contains at least three points.

We now take the disk centered at the center of that smaller square. Half its diagonal is $\frac{\sqrt{2}}{10}$, so a disk of that radius contains the whole smaller square. Also,

$$\frac{\sqrt{2}}{10} < \frac{1}{7},$$

because squaring gives $98 < 100$. Increasing the radius to $\frac{1}{7}$ still contains the three points. Thus, the required disk exists.

Problem 6.4 (Cornell Problem 4). Prove that any collection of 31 distinct integers between 1 and 60 contains two integers such that one divides the other.

Solution 6.4 (Cornell Problem 4). From [Solution 5.3](#), we know that we can express each number in the form of $a = 2^m \cdot q$, where q is odd. Between 1 and 60, there are $\frac{60}{2} = 30$ possible distinct odd values of q . Since we need to pick 31 numbers, by the Pigeonhole Principle, at least two share the same value of q . Comparing their powers of 2 shows that the smaller divides the larger.

⁸The Pigeonhole Principle Problems by Cornell Department of Mathematics

7. SET 7: PIGEONHOLE PRINCIPLE 3 ⁹

Problem 7.1 (Walk Through Combinatorics Quick Check 1). Prove that among eight integers, there are always two whose difference is divisible by seven.

Solution 7.1 (Walk Through Combinatorics Quick Check 1). There are seven possible remainders after dividing a number by seven. As there are eight integers, by Pigeonhole Principle, two numbers have the same remainder. The difference of two numbers that have the same remainder is always a multiple of the modulus. For example, when subtracting $a_2 = 7x_2 + r$ by $a_1 = 7x_1 + r$, it is evident that the remainders cancel out and the difference is $7(x_2 - x_1)$. Hence, the proof is complete.

Problem 7.2 (Walk Through Combinatorics Quick Check 2). A student wrote six distinct positive integers on the board, and pointed out that none of them had a prime factor larger than 10. Prove that there are two integers on the board that have a common prime divisor. Could we make the same conclusion if in the first sentence we replaced “six” by “five”?

Solution 7.2 (Walk Through Combinatorics Quick Check 2). Let the six distinct positive integers on the board represent the pigeons. At most one of them is 1, so at least five of them are greater than 1. Each of those five has a prime divisor, and the only possible prime divisors are 2, 3, 5 and 7. We assign each integer greater than 1 to its least prime divisor. These are four pigeonholes, so by Pigeonhole Principle, two integers share a prime divisor.

We cannot make the same conclusion for five integers. The set $\{1, 2, 3, 5, 7\}$ satisfies the given condition, but no two members have a common prime divisor. Thus, the answer to the second question is no.

Problem 7.3 (Walk Through Combinatorics Quick Check 3). A bicycle path is 30 miles long, with four rest areas. Prove that either there are two rest areas that are at most six miles from each other, or there is a rest area that is at most six miles away from one of the endpoints of the path.

Solution 7.3 (Walk Through Combinatorics Quick Check 3). We arrange the four rest areas in their order along the path. They divide the path into five gaps: one before the first rest area, three between consecutive rest areas, and one after the last rest area. The five gaps sum to 30 miles, so at least one is at most $30/5 = 6$ miles.

If that gap is between two rest areas, the first condition holds. If it is at an endpoint of the path, the second condition holds. Thus, the proof is complete.

Problem 7.4 (Walk Through Combinatorics Example 1.5a). Ten points are given within a square of unit size. Then there are two of them that are closer to each other than 0.48.

Solution 7.4 (Walk Through Combinatorics Example 1.5a). To set up the Pigeonhole Principle, we can divide the square of unit size into nine smaller congruent squares. As there are nine pigeonholes and ten pigeons, two of the ten points are forced within the same smaller square. Each of the smaller squares has a side length of $1/3$ units. To consider the worst case scenario, the two points are at most the diagonal of the smaller square apart. The distance of this diagonal can be calculated by $d = \sqrt{2} \cdot \text{side length}$. Thus, the maximum distance is $\frac{\sqrt{2}}{3} < 0.48$.

Problem 7.5 (Walk Through Combinatorics Example 1.5b). Ten points are given within a square of unit size. Then there are three of them that can be covered by a disk of radius 0.5.

Solution 7.5 (Walk Through Combinatorics Example 1.5b). To set up the Pigeonhole Principle, we can divide the square of unit size along its diagonals into four congruent triangles. As there

⁹Problems and standard solution methods adapted from A Walk Through Combinatorics by Miklos Bona Chapter 1

are four pigeonholes and ten pigeons, at least one of the triangles must contain three of the ten points. Now, we can construct a circumdisk to contain the triangle with three points. The radius of the circumcircle can be calculated using the formula $R = \frac{abc}{4k}$. The side lengths of the isosceles triangle are: 1 and half the diagonal of the unit square. Hence, 1, $\frac{\sqrt{2}}{2}$, and $\frac{\sqrt{2}}{2}$. The area is $\frac{1}{2} \cdot 1 \cdot 0.5 = 0.25$ units². $R = \frac{\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2}}{4(0.25)}$. After simplifying, this gives $\frac{0.5}{1}$. Thus, it is evident that there are three points that can be covered by a disk of radius 0.5.

Problem 7.6 (Walk Through Combinatorics Example 1.6). Consider the following family-tree model. Every parent is 25 years older than their child, every person has two distinct parents, and fewer than 2^{39} different people occur in the reader's preceding 40 generations. Prove that during those 1000 years, the reader had an ancestor A such that there was a person P who was a strict ancestor of both the father and the mother of A. Here the numerical conditions are assumptions of the model, not historical estimates.

Solution 7.6 (Walk Through Combinatorics Example 1.6). The number of ancestral positions doubles every generation going backwards. The number of different people does not have to double, because one person may occupy more than one position.

Start with one of the reader's parents. At generation 39 of that parent's pedigree there are 2^{39} positions, and they all lie in generation 40 of the reader's pedigree. By the model, fewer than 2^{39} different people are available. Hence, two of those positions must be occupied by the same person P.

Follow the two paths from the chosen parent to those positions. Let A be the last common position before the paths first separate. The two paths then go through the father and the mother of A, and both eventually reach P. Since those parents are distinct and the two occurrences of P are at the same generation, P cannot be both immediate parents of A. Thus, both paths have at least one further step, and P is a strict ancestor of both parents of A.

A is in the chosen parent's pedigree, so A is an ancestor of the reader, not the reader. Under the 25-year assumption, all the positions used lie within the preceding 1000 years. This proves the statement for the given model.

Problem 7.7 (Walk Through Combinatorics Quick Check 1b). The product of five given nonzero polynomials with real coefficients is a polynomial of degree 21. Prove that we can choose two of those polynomials so that the degree of their product is at least nine.

Solution 7.7 (Walk Through Combinatorics Quick Check 1b). It is important to note that when nonzero polynomials with real coefficients are multiplied, their degrees get summed. Let the five degrees be d_1, d_2, d_3, d_4, d_5 . Their sum is 21.

Now consider the degrees of all $\binom{5}{2} = 10$ pairwise products. Each d_i appears in four of these sums, so their total is $4 \cdot 21 = 84$. Therefore, at least one pair has degree at least $\lceil 84/10 \rceil = 9$. Hence, we can choose two of the polynomials as required.

Problem 7.8 (Walk Through Combinatorics Quick Check 2b). A college has 39 departments, and a total of 261 faculty members in those departments. Prove that there are three departments in this college whose combined number of faculty numbers is at least 21.

Solution 7.8 (Walk Through Combinatorics Quick Check 2b). Partition the total departments into thirteen pigeonholes to form groups of three departments per pigeonhole. Assign each faculty member to one department in which they work, and then to its group of three departments. This puts all 261 faculty members into those thirteen pigeonholes. If each group of three departments had at most 20 faculty members, there are 260 total faculty. However, there are 261 faculty members, which leaves at least one group of three departments with at least 21 assigned

faculty members. Therefore, there are three departments in this college whose combined number of faculty numbers is at least 21.

Problem 7.9 (Walk Through Combinatorics Quick Check 3b). Let n be a positive integer that has exactly three prime divisors, and at least seven divisors of the form p^k , where p is a prime, and k is a positive integer. Prove that n must be divisible by the cube of an integer that is larger than 1.

Solution 7.9 (Walk Through Combinatorics Quick Check 3b). Write $n = p_1^{e_1} p_2^{e_2} p_3^{e_3}$, where the three primes are distinct and the exponents are positive integers. The divisors of the form p_i^k are $p_i, p_i^2, \dots, p_i^{e_i}$, so there are $e_1 + e_2 + e_3 \geq 7$ prime-power divisors in total. Let the three primes be the pigeonholes. By Pigeonhole Principle, at least one exponent is at least $\lceil 7/3 \rceil = 3$. Thus, p_i^3 divides n for one of the primes, and the proof is complete.

Problem 7.10 (Walk Through Combinatorics Exercise 1). A busy airport sees 1500 takeoffs a day. Prove that there are two planes that take off within a minute of each other.

Solution 7.10 (Walk Through Combinatorics Exercise 1). We notice that there are $24 \cdot 60 = 1440$ minutes in a day. If we divide the 24-hour day into 1440 half-open one-minute intervals, and there are 1500 takeoffs, by the Pigeonhole Principle, one minute has two or more takeoffs. Thus, two planes take off within a minute of each other.

Problem 7.11 (Walk Through Combinatorics Exercise 2). On a particular day, ten players spent some time in a given chess club. By the end of the day, 31 games ended. Prove that there was at least one player who played at least 7 games.

Solution 7.11 (Walk Through Combinatorics Exercise 2). We notice that if 31 games were played then there were $31 \cdot 2 = 62$ player appearances in games (each game contributes two appearances). Since we have 10 people who are playing in the tournament, by the Pigeonhole Principle one player played at least $\lceil \frac{62}{10} \rceil = 7$ games and our proof is complete.

Problem 7.12 (Walk Through Combinatorics Exercise 4). We have selected 129 points inside a regular triangle of side length 100 at random. Prove that there will be three points among selected with an area of at most 68.

Solution 7.12 (Walk Through Combinatorics Exercise 4). We notice that we can divide a triangle into a perfect square number of smaller congruent triangles. It is evident that we can divide a triangle into 4 congruent triangles, 9, 16, 25 ... each of side length equal to the original side divided by 2, 3, 4 and 5, respectively. For there to be three points in one of the smaller congruent triangle we must have fewer than $129 \div 4 = 32.25$ smaller congruent triangles. To reduce the area occupied by the three points as much as possible, we want the perfect square closest to the upper bound. As 64 is a perfect square, this number works perfectly. Therefore, the side length of each smaller congruent triangle is $100 \div 4 = 25$ units. Now, by substitution into Heron's Formula, the area is $\sqrt{18.75(6.25)(6.25)(6.25)} = \frac{625\sqrt{3}}{16} < 68$, approximately 67.658. Therefore, there will be three points among selected with an area of at most 68.

Lemma 7.1 (Choosing half the total). Let $a_1, \dots, a_n$ be positive integers with sum $2n$, where every $a_i \leq n$. Then some subset of the entries sums to n , except when n is odd and every entry is 2. If at least one entry is 1, every integer from 0 to $2n$ can be made as a subset sum.

Proof. If there are no entries equal to 1, all entries are at least 2. Their sum is $2n$, so they must all be 2. We can then make n exactly when n is even.

Now suppose there are $k \geq 1$ entries equal to 1. We call these Type A entries, and call the remaining entries Type B. Write the Type B entries as $2 \leq b_1 \leq \dots \leq b_m$, where $m = n - k$.

Since the total is $2n$, we have

$$\sum_{i=1}^m (b_i - 2) = k.$$

There must be at least two Type B entries. There cannot be none, since a list of ones has sum n , and if there were only one it would equal $n + 1$, which is not allowed. Every b_i is at most $k + 2$. Also, $b_1 \leq k + 1$, because otherwise all $m \geq 2$ excesses $b_i - 2$ would be at least k , and their sum would be greater than k .

The k Type A entries let us make every integer sum from 0 to k . Suppose that some entries let us make every integer from 0 to S . Adding an entry $b \leq S + 1$ lets us make all integers from 0 to $S + b$: not using it gives $[0, S]$, and using it gives $[b, S + b]$. These two integer intervals overlap or are next to each other.

We first add $b_1 \leq k + 1$, so all sums from 0 to $k + b_1$ can be made. Now $k + b_1 \geq k + 2$, and every other $b_i \leq k + 2$. We can therefore add each remaining entry by the same argument, with no gap in the possible sums. Eventually we obtain every integer sum from 0 to $2n$, including n . $\square$

Problem 7.13 (Walk Through Combinatorics Exercise 5). We have distributed two hundred balls into one hundred boxes with the restrictions that no box got more than one hundred balls, and each box got at least one. Prove that it is possible to find some boxes that contain together exactly one hundred balls.

Solution 7.13. We call boxes with exactly one ball Type A boxes and boxes with more than one ball Type B boxes. If there are no Type A boxes, the total of 200 balls forces every box to contain exactly two balls. We then choose any 50 boxes.

Otherwise, there is a Type A box. Apply the preceding lemma to the 100 box occupancies, whose sum is 200 and each of which is at most 100. Its Type A and Type B argument shows that every integer from 0 to 200 can be made by selecting boxes. In particular, we can choose some boxes containing exactly 100 balls. Thus, the proof is complete.

Problem 7.14 (Walk Through Combinatorics Exercise 11). Prove that among 502 positive integers, there are always two integers so that either their sum or their difference is divisible by 1000.

Solution 7.14. There are 1000 possible remainders after dividing a number by 1000, ranging from 0 to 999. If two numbers have the same remainder after division, their difference is divisible by 1000. If two numbers have the complementary remainders after division, their sum is divisible by 1000. Hence, we can partition the possible remainders into pigeonholes as such: $\{0\}$, $\{1, 999\}$, $\{2, 998\}$, $\{3, 997\}$, $\{4, 996\}$... $\{499, 501\}$, $\{500\}$. This creates 499 pairwise complementary pigeonholes, plus two individual pigeonholes for residues 0 and 500. Thus, a sum of 501 pigeonholes. However, there are 502 positive integers. By Pigeonhole Principle, two integers must share the same pigeonhole so that either their sum or difference is divisible by 1000.

Lemma 7.2 (A triangle inside a rectangle). A triangle inside a rectangle of side lengths a and b has area at most $\frac{1}{2}ab$.

Proof. The area of a triangle with vertices u, v, w is $\frac{1}{2}|\det(v - u, w - u)|$. Fix two of the vertices. As the third vertex moves in the rectangle, the signed determinant is an affine function of its coordinates. The third vertex is a convex combination of the four corners, so the triangle inequality shows that its absolute determinant is no greater than the largest absolute determinant at a corner. Thus, we can move that vertex to a corner without decreasing the area.

Repeat this for the other two vertices. The resulting three corners, possibly repeated, form a triangle of area either zero or $\frac{1}{2}ab$. Therefore, the original triangle has area at most $\frac{1}{2}ab$. $\square$

Problem 7.15 (Walk Through Combinatorics Exercise 17). Sixty-five points are given inside a square of side length 1 unit. Prove that there are three of them that span a triangle of area at most $\frac{1}{32}$ square units.

Solution 7.15. We can partition the square into 25 smaller congruent squares of side length $\frac{1}{5}$. By Pigeonhole Principle, there must be at least one smaller square that contains three of the 65 given points. The area of the smaller square is $(\frac{1}{5})^2 = \frac{1}{25}$. By Lemma 7.2, the area of a triangle inside it is at most $\frac{1}{2} \cdot \frac{1}{5} \cdot \frac{1}{5} = \frac{1}{50}$. Therefore, there are three points that span a triangle of area at most $\frac{1}{32}$ square units.

Problem 7.16 (Walk Through Combinatorics Exercise 19). A state has seven counties. In one year, three candidates run in a statewide election. Is it possible that in each county the same number of people vote, and the candidate who gets the highest number of votes statewide does not get the highest number of votes in any county?

Solution 7.16. Let Candidate B be the candidate who gets the highest number of votes statewide, but does not get the highest number of votes in any county. We can give one example with ten voters in every county.

In four counties, let the votes for Candidates A, B and C be 5, 4 and 1, respectively. In the other three counties, let their votes be 1, 4 and 5. Every county has ten votes, and B does not have the highest total in any county. Statewide, the totals are

$$A : 4 \cdot 5 + 3 \cdot 1 = 23, \quad B : 7 \cdot 4 = 28, \quad C : 4 \cdot 1 + 3 \cdot 5 = 19.$$

Thus, B has the highest statewide total while losing every county, so it is possible.

Problem 7.17 (A Walk Through Combinatorics Exercise 20). Prove that every year contains at least 4 and at most 5 months which contain 5 Sundays.

Solution 7.17 (A Walk Through Combinatorics Exercise 20). We notice that every month contains at least 28 days, or 4 weeks. This means that Sunday occurs at least 4 times each month. We also know that every month contains at most 31 days which means that at most each month can contain Sunday 5 times. We also notice that there are 365 or 366 days in a year, which is more than 52 weeks in an year, which means that at least 52 Sundays in the year. We also know that there are less than 53 weeks in a year which means that Sunday can occur at most 53 times in a year. Now, let us combine all of this information.

We know that 4 Sundays happen each month at least which means 48 Sundays over the span of an year. However, we need to have 52 or 53 Sundays in an year which requires an additional 4 – 5 Sundays depending on the year. We also know that since a month cannot have more than 5 Sundays that it can add at most 1 to the existing total (since 4 Sundays have already been counted). Thus, if there are 52 Sundays in the year, we have 4 months with 5 Sunday which 'add 1' to 48 Sundays. Otherwise, if there are 53 Sundays in the year, we have 5 months with 5 Sundays since $53 - 48 = 5$. Therefore, our proof is complete.

Problem 7.18 (A Walk Through Combinatorics Exercise 21). A soccer league features 17 games for today. Let us assume that no team will score more than 3 goals. Prove that there will be a result that will occur more than once. Note: a result consists of the number of goals scored by the home team followed by number of goals scored by away team. Therefore, 2-3 and 3-2 are different results.

Solution 7.18 (A Walk Through Combinatorics Exercise 21). We notice that both the home team and away team have 4 possible choices for number of goals scored: 0, 1, 2 or 3. Thus, there is a total of 16 possible results and 17 total games. Therefore, by the Pigeonhole Principle, at least one result has to be repeated.

Problem 7.19 (A Walk Through Combinatorics Exercise 22). A group of seven co-workers are trying to predict the total number of points in a given basketball game. The first six people already took their guesses, and, curiously, they all picked six distinct even numbers. Mr. Slow is the last person to guess, and he knows all previous guesses. Is there a strategy for him to guess better than half of his colleagues.

Consider both meanings: closer to the actual score than at least three colleagues, or closer than at least four colleagues.

Solution 7.19 (A Walk Through Combinatorics Exercise 22). We arrange the six previous guesses as $g_1 < g_2 < \dots < g_6$. Mr. Slow can pick an odd integer t between g_3 and g_4 , because these are distinct even integers.

Let x be the actual score. If $x \leq t$, then each of g_4, g_5, g_6 is farther from x than t is. If $x \geq t$, then each of g_1, g_2, g_3 is farther from x than t is. Thus, he is guaranteed to be strictly closer than at least three of his colleagues.

However, there is no general guarantee of being closer than four colleagues. For example, take the guesses 2, 4, 6, 8, 10, 12. To beat four of them when the actual score is 0, we would need $|t| < 6$. To beat four when the actual score is 100, we would need $|100 - t| < 92$, which implies $t > 8$. These conditions cannot both hold. Therefore, the first interpretation has a strategy, but the stronger interpretation does not.

Problem 7.20 (A Walk Through Combinatorics Exercise 23). A soccer team scored a total of 40 goals this season. Nine players scored at least one goal. Prove that at least two players scored the same number of goals.

Solution 7.20 (A Walk Through Combinatorics Exercise 23). Let Player 1 score the least number of goals and so on up until Player 9, who scores the most amount of goals. Now, we will prove via contradiction and try to see if we can meet the requirements. If the nine positive integer counts are all distinct and ordered, Player i must have scored at least i goals. Thus, the smallest possible counts would be 1, 2, ..., 9. Then, we have a total of at least $1 + 2 + 3 + 4 + \dots + 9 = 45$ goals scored, which is above 40 goals. If each player scores a distinct number of goals, we go above the 40 goal target. Thus, to score 40 goals at least two players have to score the same number of goals.

Problem 7.21 (A Walk Through Combinatorics Exercise 25a and 25b). We select 11 distinct positive integers at random that are less than 29. Prove that there will always be two integers selected that have a common divisor greater than 1. Also, would this be true if we only chose 10 distinct integers

Solution 7.21 (A Walk Through Combinatorics Exercise 25a and 25b). We use a proof by contradiction. Suppose the selected integers are pairwise coprime. We can include 1 at most once. For each remaining integer, choose its least prime divisor. Below 29, the possible prime divisors are

$$2, 3, 5, 7, 11, 13, 17, 19, 23.$$

We can have at most one selected integer for each of these nine primes, because two assigned to the same prime would have a common divisor greater than 1. Thus, we can choose at most $1 + 9 = 10$ pairwise coprime integers. This proves that 11 distinct integers force the required pair.

For 10 integers the conclusion need not hold: $\{1, 2, 3, 5, 7, 11, 13, 17, 19, 23\}$ is a pairwise coprime set. Therefore, the answer to the second part is no.

Problem 7.22 (A Walk Through Combinatorics Exercise 27). The sum of five positive real numbers is 100. Prove that there are two numbers among them whose difference is at most 10.

Solution 7.22 (A Walk Through Combinatorics Exercise 27). We can solve this problem by proving via contradiction. Arrange the five numbers as $0 < x_1 \leq x_2 \leq \dots \leq x_5$. Assume that no two have a difference at most 10. Then every consecutive gap is greater than 10, so

$$x_2 > x_1 + 10, \quad x_3 > x_1 + 20, \quad x_4 > x_1 + 30, \quad x_5 > x_1 + 40.$$

Adding these inequalities and x_1 gives

$$100 = x_1 + x_2 + x_3 + x_4 + x_5 > 5x_1 + 100 > 100.$$

This is a contradiction. Thus, two of the numbers have a difference at most 10.

Problem 7.23 (A Walk Through Combinatorics Exercise 30). We are given 17 points inside a regular triangle of side length one unit. Prove that there are two points among them whose distance is not more than $\frac{1}{4}$ unit.

Solution 7.23 (A Walk Through Combinatorics Exercise 30). Partition the triangle of side length one unit into sixteen congruent smaller triangles of side length $\frac{1}{4}$ unit. As there are sixteen pigeonholes and 17 points inside the regular triangle, by Pigeonhole Principle, there is one triangle that contains two points. For any triangle, the maximum distance is always the distance between two of its vertices. Therefore, there are two points among them whose distance is not more than $\frac{1}{4}$ unit.

Problem 7.24 (A Walk Through Combinatorics Exercise 31). Prove the sequence

$$1967, \quad 19671967, \quad 196719671967, \quad \dots$$

contains an element divisible by 1969.

Solution 7.24 (A Walk Through Combinatorics Exercise 31). Let B_k be the integer consisting of k copies of the block 1967. Appending one block gives $B_{k+1} = 10^4 B_k + 1967$.

Now suppose none of $B_1, B_2, \dots, B_{1969}$ is divisible by 1969. These 1969 numbers have only 1968 possible nonzero remainders, so two, say B_i and B_j with $i < j$, have the same remainder. Their difference satisfies

$$B_j - B_i = 10^{4i} B_{j-i}.$$

Since $\gcd(10^4, 1969) = 1$, the divisibility of the difference by 1969 implies that B_{j-i} is divisible by 1969. This contradicts the assumption, because $1 \leq j - i \leq 1968$. Thus, the sequence contains an element divisible by 1969.

Problem 7.25 (A Walk Through Combinatorics Exercise 34). We select $n + 1$ different integers from the set $\{1, 2, \dots, 2n\}$. Prove that there will always be two among the selected integers whose greatest common divisor is 1.

Solution 7.25 (A Walk Through Combinatorics Exercise 34). We can prove this problem via contradiction. We select $n + 1$ different integers from the set $\{1, 2, \dots, 2n\}$, so that no two among the selected integers have a greatest common factor of 1. As any two consecutive numbers are forced to have a greatest common factor of 1, we can now establish our pigeonholes based this observation. This results in the following parity configuration: $\{2j - 1, 2j\}$ for $j = 1, 2, \dots, n$. As all $2n$ numbers are paired with its consecutive number, there are n pigeonholes. However, we must select $n + 1$ different integers from the set, meaning two integers are drawn from the same pigeonhole. Thus, there will always be two among the selected integers whose greatest common divisor is 1.

Problem 7.26 (A Walk Through Combinatorics Exercise 35 a). Let $n \geq 2$. We select $n + 1$ different integers from the set $\{1, 2, \dots, 2n\}$. Is it true that there will always be two among the selected integers so that one of them is equal to twice the other?

Solution 7.26 (A Walk Through Combinatorics Exercise 35 a). No, this is not always true. Let $n = 3$ and select the four integers $\{1, 3, 4, 5\}$ from $\{1, 2, 3, 4, 5, 6\}$. The double of 1 is not selected, and the doubles of 3, 4 and 5 are not selected either. Thus, there is no pair with one equal to twice the other.

The odd-part pigeonholes from [Solution 5.3](#) are chains such as $\{1, 2, 4\}$, not disjoint pairs. Two selected numbers in one chain may have a ratio of 4 or a larger power of 2 instead of exactly 2. This is why the stronger conclusion fails.

Problem 7.27 (A Walk Through Combinatorics Exercise 35 b). Is it true that there will always be two among the selected integers so that one is a multiple of the other?

Solution 7.27 (A Walk Through Combinatorics Exercise 35 b). Yes. By the odd-part construction in [Solution 5.3](#), every selected integer can be written as $2^e q$ with q odd. There are only n possible odd parts, namely $1, 3, \dots, 2n - 1$. Therefore, two of the $n + 1$ distinct selected integers have the same odd part. The larger divided by the smaller is a positive integral power of 2, so one is a multiple of the other. Unlike the previous question, this does not require the ratio to be exactly 2.

8. SET 8: PIGEONHOLE PRINCIPLE 4 ¹⁰

Problem 8.1 (Example 3.2.1). For this exercise, assume every person is assigned to one of the two categories called male and female. Among any group of seven people, there must be at least 4 of the same sex.

Solution 8.1 (Example 3.2.1). Let the two categories in the problem represent the pigeonholes. Let the seven people represent the pigeons. By Pigeonhole Principle, one pigeonhole contains at least $\lceil \frac{7}{2} \rceil = 4$ people.

Problem 8.2 (Example 3.2.2). Among any group of 13 people, there must be at least two whose birthdays are in the same month.

Solution 8.2. There are 12 possible months in a year for a birthday to fall, which represents the pigeonholes. Let the 13 people in the group represent the pigeons. By Pigeonhole Principle, at least two people must share the same pigeonhole. Therefore, a minimum of two people among the group have birthdays that are in the same month.

Problem 8.3 (Example 3.2.3). Among any group of 3000 people, there are at least 9 who have the same birthday.

Solution 8.3 (Example 3.2.3). Let the at most 366 possible month-and-day birthdays, including February 29, represent the pigeonholes. Let the 3000 people represent the pigeons. By Pigeonhole Principle, note that at least $\lceil \frac{3000}{366} \rceil = 9$ people must fall into the same pigeonhole. Thus, the condition is met.

Problem 8.4 (Example 3.2.4). Show that for any set of 10 points chosen within a square whose sides are of length 3 units, there are two points in the set whose distance apart is at most $\sqrt{2}$.

Solution 8.4 (Example 3.2.4). Divide the square of side length three into nine smaller congruent squares of side length 1 unit. By Pigeonhole Principle, at least two points must fall into the same square. The maximum distance between the two points is the diagonal of the smaller square. Using the properties of special triangles, we can construct a 1-1- $\sqrt{2}$ triangle, where the hypotenuse is the diagonal of the smaller square. Therefore, there are two points in the set whose distance apart is at most $\sqrt{2}$ units.

Problem 8.5 (Example 3.3.1). Let $A = \{a_1, a_2, \dots, a_5\}$ be a set of five positive integers. Show that for any permutation $a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}, a_{i_5}$ of A , the product $(a_{i_1} - a_1)(a_{i_2} - a_2) \cdots (a_{i_5} - a_5)$ is always even.

Solution 8.5 (Example 3.3.1). If one factor is even, the product of the five factors is even as well. By Pigeonhole Principle, at least three of the five numbers have the same parity. Choose three original positions occupied by numbers of this parity. There are at most two numbers of the opposite parity in the whole set.

For all three differences at these positions to be odd, the permutation would have to put a number of the opposite parity into each of these three positions. This is impossible because there are at most two such numbers. Thus, at least one position pairs two numbers of the same parity. Their difference is even, and the product is always even.

Problem 8.6 (Example 3.3.2). Ten players took part in a round robin chess tournament (ie., each player must play exactly one game against every other player). According to the rules, a player scores 1 point if he wins a game; -1 point if he loses; and 0 points if the game ends in a draw. When the tournament was over, it was found that more than 70% of the games ended in a draw. Show that there were two players who had the same total score.

¹⁰Principles and Techniques in Combinatorics by Chen Chuan-Chong, et. al Ch3

Solution 8.6 (Example 3.3.2). We must begin by calculating the total number of games played at the round robin chess tournament. Since every player plays exactly once against every other player, there are $\binom{10}{2} = 45$ games. More than 70% of them were draws, so at least 32 were draws and at most 13 were wins or losses.

Now assume that all ten players have different scores. At most one player has score 0, so at least five players have positive scores or at least five have negative scores. If five have positive scores, their scores sum to at least $1 + 2 + 3 + 4 + 5 = 15$. Each player's positive score is at most his number of wins, so at least 15 games must have had a winner. If five have negative scores, the same argument with losses gives at least 15 games with a loser. Either case contradicts the limit of 13 games not ending in draws. Therefore, two players have the same total score.

Problem 8.7 (Example 3.3.3). Let A be a set of m positive integers where $m \geq 1$. Show that there exists a nonempty subset B of A such that the sum $\sum_{x \in B} x$ is divisible by m .

Solution 8.7 (Example 3.3.3). We put the elements of A in an order, $A = \{a_1, a_2, \dots, a_m\}$, and look at the m prefix sums $S_j = a_1 + a_2 + \dots + a_j$ for $1 \leq j \leq m$. There are m possible remainders upon division by m .

If one of the prefix sums has remainder 0, we have found our nonempty subset. Otherwise, these m sums have only $m - 1$ possible remainders. By the Pigeonhole Principle, two sums have the same remainder, say S_i and S_j with $i < j$. Then

$$S_j - S_i = a_{i+1} + a_{i+2} + \dots + a_j$$

is divisible by m . Since $j > i$, these entries form a nonempty subset of A . Thus, our proof is complete.

Problem 8.8 (Example 3.3.4). Let $X \subseteq \{1, 2, \dots, 99\}$ and $|X| = 10$. Show that it is possible to select two disjoint nonempty proper subsets Y, Z of X such that $\sum_{y \in Y} y = \sum_{z \in Z} z$.

Solution 8.8 (Example 3.3.4). The total number of nonempty proper subsets that can be formed from ten numbers is $2^{10} - 2 = 1022$ because each number has the choice of being included or excluded from the subset. The sum of any proper subset of $X \subseteq \{1, 2, \dots, 99\}$ is at most $91 + 92 + 93 + 94 + 95 + 96 + 97 + 98 + 99 = 855$. Hence, there are at most 855 possible sums for the 1022 nonempty proper subsets Y, Z of X . By Pigeonhole principle, there must exist at least two subsets A, B that share the same sum. However, subsets A and B are not necessarily disjoint as they are any two subsets of X , meaning they could share certain elements. To force the two subsets to become disjoint, we must remove the elements they have in common, or the intersection of the two sets. Since we already established that the $\sum_{a \in A} a = \sum_{b \in B} b$, and the intersection that was removed is constant in both sets, the remaining elements in the two disjoint sets must have an equal sum. This results in $\sum_{y \in Y} y = \sum_{z \in Z} z$, which completes the proof.

Neither remaining subset can be empty: if one were empty, equality of sums and positivity of the entries would make the other empty too, which would mean $A = B$. Since the two remaining subsets are disjoint and nonempty, neither can be all of X . Thus, they are also proper subsets.

Problem 8.9 (Example 3.4.1). Seventeen people correspond by email with one another - each one with all the rest. In their letters only three different topics are discussed. Each pair of correspondents deals with only one of these topics. Prove that there are at least three people who write to one another about the same topic.

Solution 8.9 (Example 3.4.1). We convert the 17 people into 17 points and connect each pair of points with each other in one of three colors: red, blue and yellow. Now, if we have a triangle of the same color, then we have three people writing to each other about the same topic. Now, we take the first point and notice it has 16 edges coming out from it, one to each of

the other points. By the Pigeonhole Principle, at least $\lceil \frac{16}{3} \rceil = 6$ of these edges have to be of one color, which is blue WLOG. Now, let us assume that the first point, P_1 has a blue edge with $P_2, P_3, P_4, P_5, P_6, P_7$. Now, we notice that if any of the edges between these 6 points are blue, then we have a monochromatic triangle. Otherwise, if none of the edges are blue, then they have to be either yellow or red. Now, the six points form a complete graph with only red and yellow edges. By the argument in Solution 8.11, there is a red or yellow triangle. Thus, our proof is complete.

Problem 8.10 (Example 3.4.2). Prove that at a gathering of any six people, some three of them are either mutual acquaintances or complete strangers to one another.

Assume acquaintance is symmetric, and each pair either knows each other or does not.

Solution 8.10 (Example 3.4.2). Let the six people of the gathering be six points in space. Let the 15 line segments connecting each of the six points be colored red if they are acquaintances, or blue if they are complete strangers. This sets up the exact same question as Solution 8.11.

Problem 8.11 (Example 3.4.3). Six points are in general position in space (no three in a line, no four in a plane). The fifteen line segments joining them in pairs are drawn, and then painted with some segments red and the rest blue. Prove that some triangle has all its sides the same colour.

Solution 8.11 (Example 3.4.3). The fifteen line segments joining the six vertices are either red or blue. Therefore, if we focus a specific vertex, say A , the five line segments joining A with the other points in space are colored either red or blue. By Pigeonhole Principle, at least three lines must be of one colour. WLOG, suppose these three line segments are AB , AC and AD and they are red in colour. This means if any of BC , BD or CD are colored red, we have a monochromatic triangle ABC , ABD , ACD , respectively. However, should none of BC , BD or CD be red, we now have a monochromatic triangle of colour blue. Therefore, with six points in space connected by red or blue line segments, a monochromatic triangle will always be formed.

Problem 8.12 (Exercise 1). Show that among 5 points in an equilateral triangle of unit side length, there are 2 whose distance is at most $\frac{1}{2}$ unit apart.

Solution 8.12 (Exercise 1). Partition the equilateral triangle of unit side length into four smaller congruent triangles of side length $\frac{1}{2}$. By Pigeonhole Principle, there are four holes and five points in the triangle, which forces at least one hole to contain two points. Since the maximum distance between two points contained in a equilateral triangle is the side length of the triangle, it is evident are 2 whose distance is at most $\frac{1}{2}$ unit apart.

Problem 8.13 (Exercise 2). Given any set C of $n + 1$ distinct points ($n \in \mathbb{N}$, $n \geq 2$) on the circumference of a unit circle, show that there exist $a, b \in C$, $a \neq b$, such that the distance between them does not exceed $2 \sin \frac{\pi}{n}$.

Solution 8.13 (Exercise 2). Split the circle into n equal arcs, assigning each dividing endpoint to just one arc. Since there are $n + 1$ points, by pigeonhole principle, at least two points fall in the same arc. Each arc has angle $2\pi/n \leq \pi$. Thus, the angle between those two points is at most $2\pi/n$, and their chord length is at most $2 \sin(\pi/n)$. The condition is met.

Problem 8.14 (Exercise 3). Given any set S of 9 points within a unit square, show that there always exist 3 distinct points in S such that the area of the triangle formed by these three points is less than or equal to $\frac{1}{8}$.

Solution 8.14. Partition the square of unit length into four smaller congruent squares by connecting the midpoints of opposite sides of the square. Each smaller congruent square has a side length of $\frac{1}{2}$ unit. By Pigeonhole Principle, there must one smaller square that contains $\lceil \frac{9}{4} \rceil = 3$

points. By Lemma 7.2, every triangle inside that square has area at most half the area of the square, which is $\frac{1}{2}(\frac{1}{2})^2 = \frac{1}{8}$. Therefore, the maximum area of the triangle formed by these three points is less than or equal to $\frac{1}{8}$.

Problem 8.15 (Exercise 4). Show that given any set of five integers, there are 3 numbers in the set whose sum is divisible by three.

Solution 8.15. After dividing a number by three, the three possible remainders are 0, 1, and 2. For three numbers to have a sum that is divisible by three, they must fulfill either of the two following cases.

Case 1: They must all have the same residue. This works because $0 + 0 + 0 \equiv 0 \pmod{3}$, $1 + 1 + 1 \equiv 0 \pmod{3}$ and $2 + 2 + 2 \equiv 0 \pmod{3}$. If there are no three numbers that have different residues, by Pigeonhole principle, there are two holes and five numbers. Thus, there must be three numbers that share the same pigeonhole and whose sum is divisible by three.

Case 2: They must all have different residues. It is clear that when all three numbers have a different residues, then $0 + 1 + 2 \equiv 0 \pmod{3}$. If there are no three numbers that have the same residue, each Pigeonhole can carry a maximum of two numbers. Therefore, every single pigeonhole must contain at least one number.

As one of the two cases is always guaranteed, there are 3 numbers in the set whose sum is divisible by three.

Problem 8.16 (Exercise 5). Let $A \subseteq \mathbb{Z}$ be a set of $n + 1$ integers, where $n \geq 1$ is an integer. Show that there exist $a, b \in A$ with $a \neq b$ such that $n \mid (a - b)$.

Solution 8.16. Take each element in $A \pmod{n}$. Note that by pigeonhole principle, at least 2 elements must be the same mod n . Therefore, their difference must be divisible by n .

Problem 8.17 (Exercise 7). Let $n \geq 1$ be an integer, $A \subseteq \{1, 2, \dots, 2n\}$ and $|A| = n + 1$. Prove that there exists two distinct elements a, b in Set A such that $a \mid b$.

Solution 8.17 (Exercise 7). From Solution 5.3, it is evident that every number can be written in the form of $a = 2^m \cdot q$, with q being of odd parity. From 1 to $2n$, there are n distinct possible values of q . However, there are $n + 1$ elements in Set A . Thus, by the Pigeonhole Principle, there exists two elements a, b in Set A such that $a \mid b$.

Problem 8.18 (Exercise 9). Show that among any group of n people where $n \geq 2$, there are at least 2 people who know the same number of people, where knowing is a symmetrical relationship.

Here we count only other people, not the person himself.

Solution 8.18 (Exercise 9). We notice that the least number of people a person could know is 0 and the most a person could know is $n-1$, which is if they know all other people. That is a total of n possible values of knowing people. We will prove via contradiction. We define the number of people Person X knows to be P_x . This also means that the only way for no two people to know the same number of people is if P_x is different for $1 \leq x \leq n$ and each value from 0 to $n-1$ has a bijection with the values of P_x . This means that one person knows 0 people while another person knows $n-1$ people. However, this contradicts our argument because if a person knows $n-1$ people then they also know the person who knows 0 people, but since knowing is a symmetric relationship, this contradicts itself. Thus, our proof is complete and there are at least 2 people who know the same number of people.

Problem 8.19 (Exercise 10). Let $n \geq 1$ be an integer. Let $C = \{r_1, r_2, \dots, r_{n+1}\}$ be a set of $n+1$ real numbers, where $0 < r_i < 1$ for each $i = 1, 2, 3, \dots, n+1$. Show that there exist r_p, r_q in C , where $p \neq q$ such that $|r_p - r_q| < \frac{1}{n}$.

Solution 8.19 (Exercise 10). Arrange the set of $n+1$ real numbers in a line of length smaller than one in an increasing order, relabeling the entries in that order. Let p and q be consecutive numbers, such that $p = q - 1$. When arranging the numbers in such way, this leaves n gaps. The average length of each gap is less than $\frac{1}{n}$. By contradiction, if every gap is of length $\frac{1}{n}$ or greater, the total length of the line would be greater than or equal to one. Hence, it is impossible to make all line segments greater than $\frac{1}{n}$ because C is a set of real numbers where $0 < r_i < 1$. Thus, there exists r_p, r_q in C , such that $|r_p - r_q| < \frac{1}{n}$.

Problem 8.20 (Exercise 13). Let there be given nine distinct lattice points (points with integral coordinates) in the three dimensional Euclidean space. Show that there is a lattice point on the interior of one of the line segments joining two of these points.

Solution 8.20 (Exercise 13). Each lattice point has three integer coordinates. Every coordinate is either even or odd, so there are $2^3 = 8$ possible parity triples. Let these triples be the pigeonholes. Among nine distinct points, two points must have the same parity in all three coordinates.

The midpoint of these two points has integer coordinates, because each coordinate is the average of two integers of the same parity. Since the two points are distinct, their midpoint lies strictly inside the segment joining them. Thus, there is a lattice point in the interior of one of the segments, as required.

Problem 8.21 (Exercise 14 - Part 1). A point (a_1, a_2) is called a lattice point if both a_1 and a_2 are integers. Given any set L_2 of 5 lattice points in the x - y plane, show that there exist 2 distinct members in L_2 whose midpoint is also a lattice point (not necessarily in L_2).

Solution 8.21 (Exercise 14 - Part 1). In order to obtain an integer midpoint, the two points must have x -coordinates and y -coordinates of the same parity because this ensures their sum is a multiple of two. There are four possible combinations of parities: (even, even), (even, odd), (odd, even), and (odd, odd). By Pigeonhole Principle, there must be at least two points that share the same parity of coordinates. Thus, there exists two distinct members in L_2 whose midpoint is also a lattice point.

Problem 8.22 (Exercise 14 - Part 2). A point $(a_1, a_2, \dots, a_n)$ in the space R^n , where $n \geq 2$ is an integer, is called a lattice point if all the a_i 's are integers. Show that given any set of L_n of $2^n + 1$ lattice points in R^n , there exists two distinct members in L_n whose midpoint is also a lattice point (not necessarily in L_n).

Solution 8.22 (Exercise 14 - Part 2). For every point, there are two choices for the parity of each coordinate, even or odd, and n positions in the space R^n . Thus, in n dimensions, there are 2^n combinations for the parities. Since there are $2^n + 1$ lattice points, by Pigeonhole Principle, two points will share the same parity of coordinates. Therefore, by the number theory in Part 1 of the problem, there exists two distinct members in L_n whose midpoint is also a lattice point.

Problem 8.23 (Exercise 22). In a football tournament of one round (each team plays each other once, 2 points for win, 1 point for draw and 0 points for loss), 28 teams compete. During the tournament, more than 75% of the matches finished in a draw. Prove that there were two teams who finished with the same number of points.

Solution 8.23 (Exercise 22). Since every team plays each other once, there are $\binom{28}{2} = 378$ games. More than 75% were draws, so at least 284 were draws and at most 94 were decisive games.

Every team plays 27 games. We can subtract one point per game from every team's score, changing the points for a win, draw and loss to $+1, 0, -1$. This subtracts the same total of 27 from each team, so it does not change whether two totals are equal.

Now assume all the new scores are distinct. At most one is zero, so at least 14 scores have the same sign. Their different positive magnitudes sum to at least $1 + 2 + \cdots + 14 = 105$. If their sign is positive, these teams must have at least 105 wins in total; if it is negative, they must have at least 105 losses. Each decisive game has only one winner and one loser, so either case requires at least 105 decisive games. This contradicts the limit of 94. Therefore, two teams finish with the same number of points.

Problem 8.24 (Exercise 23). Fifteen problems, numbered 1 through 15, are posed on a certain examination. No student answers two consecutive problems correctly. If 1600 candidates sit the test, must at least two of them have the identical answer patterns.

An answer pattern records only which questions are correct and which are incorrect, not the actual written answers.

Solution 8.24 (Exercise 23). Let A_j be the number of possible correctness patterns on a test with j questions, where 1 means correct and 0 means incorrect. Note that if the last answer is incorrect, the first $j - 1$ answers can be any allowed pattern. If the last answer is correct, the preceding answer must be incorrect, and the first $j - 2$ answers can be any allowed pattern. Therefore,

$$A_j = A_{j-1} + A_{j-2} \quad (j \geq 2), \quad A_0 = 1, \quad A_1 = 2.$$

With the convention $F_0 = 0, F_1 = 1$ for Fibonacci numbers, this gives $A_j = F_{j+2}$. In particular, $A_{15} = F_{17} = 1597$. Since 1600 candidates sit the test and there are only 1597 possible patterns, by Pigeonhole Principle, at least two students have identical correctness patterns.

Problem 8.25 (Exercise 24). Let $n \geq 1$ be an integer. Suppose that $a_1 \leq a_2 \leq \cdots \leq a_n$ are positive integers such that

$$a_1 + \cdots + a_n = 2n$$

and $a_n \neq n + 1$. Show that if n is even, then for some subset $K \subset \{1, 2, \dots, n\}$ it is true that

$$\sum_{i \in K} a_i = n.$$

Show that this is true also if n is odd when we make the additional assumption that $a_n \neq 2$.

Solution 8.25. Since all n entries are positive and their sum is $2n$, the largest one satisfies $a_n \leq 2n - (n - 1) = n + 1$. The hypothesis $a_n \neq n + 1$ therefore gives $a_n \leq n$.

If n is even, Lemma 7.1 now gives a subset of the entries whose sum is n . If n is odd, the only exception in that lemma is the list with every entry equal to 2. Since the entries are positive with average 2, this is exactly the case $a_n = 2$. The additional hypothesis excludes it, so the same lemma gives the required subset when n is odd as well.

The selected entries can be identified by their indices, giving a subset K with $\sum_{i \in K} a_i = n$. The subset is nonempty and proper because all entries are positive and the full sum is $2n$.

Both exclusions matter: the list $(1, \dots, 1, n + 1)$ has no subset sum n , and an odd number of 2s has no odd subset sum.

Problem 8.26 (Exercise 25). Let X be a nonempty set having n elements and C be a colour set with $p \geq 1$ elements. Find the greatest number p such that the following property holds: If we colour in an arbitrary way each subset of X with colours from C such that each subset receives only one colour, then there exist two distinct subsets A, B of X such that the sets

$A, B, A \cup B, A \cap B$ have the same colour.

Only A and B are required to be distinct; the four named sets need not all be distinct.

Solution 8.26. We claim that the largest possible value is $p = n$.

First, order the elements as $X = \{x_1, x_2, \dots, x_n\}$. Consider the $n + 1$ subsets

$$\emptyset, \{x_1\}, \{x_1, x_2\}, \dots, X.$$

If we use at most n colours, two of these subsets have the same colour. Call them A and B with $A \subset B$. Then $A \cap B = A$ and $A \cup B = B$, so all four named sets have the same colour.

Now use $n + 1$ colours, one for each possible cardinality from 0 to n , and colour each subset by its cardinality. If $A, B, A \cup B, A \cap B$ all had the same colour, we would have $|A \cap B| = |A| = |B|$. Since $A \cap B$ is contained in both A and B , this would force $A = B$, which is not allowed. Thus, $n + 1$ colours can avoid the condition. The same colouring is available when more colours are allowed, so the maximum is n .

Problem 8.27 (Exercise 26). Consider the homogeneous system of $p \geq 1$ equations in $q = 2p$ unknowns

$$\sum_{j=1}^q a_{ij}x_j = 0 \quad (1 \leq i \leq p), \quad a_{ij} \in \{-1, 0, 1\}.$$

Prove that the system has a solution $(x_1, \dots, x_q)$ such that all x_j are integers, at least one $x_j \neq 0$, and $|x_j| \leq q$ for all j .

Solution 8.27. Let A be the coefficient matrix. We choose vectors $t = (t_1, \dots, t_q)$ with every t_j in $\{0, 1, \dots, q\}$ and compare their image vectors At . There are $(q + 1)^q$ choices for t .

In row i , let u_i be the number of coefficients equal to 1 and let v_i be the number equal to -1 . The entry $(At)_i$ is an integer between $-qv_i$ and qu_i . Thus, it has at most

$$q(u_i + v_i) + 1 \leq q^2 + 1$$

possible values. There are therefore at most $(q^2 + 1)^p$ possible image vectors. But, since $q = 2p$,

$$(q + 1)^q = ((q + 1)^2)^p > (q^2 + 1)^p.$$

By the pigeonhole principle, two different vectors t and s have the same image. Let $x = t - s$. Then $Ax = 0$, and x is not the zero vector because $t \neq s$. Each coordinate is an integer difference of two numbers in $\{0, 1, \dots, q\}$, so $|x_j| \leq q$. This gives every required condition.

Problem 8.28 (Exercise 27). An international society has its members from six different countries. The list of members contains 1978 names, numbered $1, 2, \dots, 1978$. Prove that there is at least one member whose number is either the sum of the numbers of two members from his own country, or twice the number of one member from his own country.

Solution 8.28 (Exercise 27). We use a graph-colouring argument, extending the idea from Solution 8.11. A complete graph has an edge between every pair of different vertices. Define $T_1 = 3$, and for $r \geq 2$ define

$$T_r = r(T_{r-1} - 1) + 2.$$

We first show by induction that, if the edges of a complete graph on T_r vertices have r colours, some triangle has all its edges the same colour.

For $r = 1$, the three vertices give the triangle. Now assume the result for $r - 1$ colours. Fix one vertex in the graph on T_r vertices. It has $r(T_{r-1} - 1) + 1$ edges coming out, so at least T_{r-1} of them have one colour. If two of the corresponding neighbours have an edge of this colour between them, we have our triangle. Otherwise, all edges among those neighbours have only the

other $r - 1$ colours, and the induction hypothesis gives a triangle there. Thus, the claim holds. Computing these bounds gives

$$T_1 = 3, T_2 = 6, T_3 = 17, T_4 = 66, T_5 = 327, T_6 = 1958.$$

Now take vertices numbered $0, 1, \dots, 1978$. For $u < v$, colour the edge between u and v according to the country of member $v - u$. There are $1979 \geq 1958$ vertices and six colours, so the claim gives a triangle with vertices $u < v < w$ whose edges all have one colour. Thus, the numbers

$$a = v - u, \quad b = w - v, \quad c = w - u$$

belong to the same country, and $a + b = c$. If $a \neq b$, the number of member c is the sum of the numbers of two different members from his country. If $a = b$, it is twice the number of member a . In both cases the required condition holds.

Problem 8.29 (Exercise 28). Let $p, q \in \mathbb{N}$. Show that in any sequence of $R(p, q)$ distinct integers, there is either an increasing subsequence of p terms or a decreasing subsequence of q terms.

Here $R(p, q)$ is the least number of vertices such that every red-blue colouring of the edges of a complete graph contains either p vertices with all their joining edges red or q vertices with all their joining edges blue.

Solution 8.29 (Exercise 28). If $p = 1$ or $q = 1$, a one-term subsequence gives the required conclusion. Otherwise, let the sequence be $z_1, \dots, z_{R(p,q)}$. We use its positions as the vertices of a complete graph. For $i < j$, colour the edge red if $z_i < z_j$, and blue if $z_i > z_j$. The values are distinct, so every edge has one of these two colours.

By the definition of $R(p, q)$, there is a red group of p vertices or a blue group of q vertices. Reading their positions in increasing order gives an increasing subsequence in the first case and a decreasing subsequence in the second. Thus, the proof is complete.

A further observation. The longest-subsequence argument gives a stronger length bound of $(p - 1)(q - 1) + 1$ for $p, q \geq 2$. This is not being identified with $R(p, q)$.

For each position i in the sequence, define:

a_i = length of the longest increasing subsequence ending at i ,

b_i = length of the longest decreasing subsequence ending at i .

Each pair (a_i, b_i) satisfies

$$1 \leq a_i \leq p - 1, \quad 1 \leq b_i \leq q - 1$$

if there is no increasing subsequence of length p and no decreasing subsequence of length q .

Thus there are at most

$$(p - 1)(q - 1)$$

possible pairs (a_i, b_i) .

If the sequence has more than $(p - 1)(q - 1)$ elements, then by the pigeonhole principle two indices $i < j$ have the same pair (a_i, b_i) .

But comparing the two values forces either $a_j > a_i$ or $b_j > b_i$, a contradiction.

Therefore any sequence of length

$$(p - 1)(q - 1) + 1$$

must contain either an increasing subsequence of length p or a decreasing subsequence of length q .

Problem 8.30 (Exercise 29). Show that given any sequence of $pq + 1$ distinct real numbers, where p and q are nonnegative integers, there is either an increasing subsequence of $p + 1$ terms or a decreasing subsequence of $q + 1$ terms.

Solution 8.30 (Exercise 29). If $p = 0$ or $q = 0$, the sequence has one term, which is already the required subsequence of length one. Now suppose $p, q \geq 1$. For each position i define:

$a_i = \text{length of the longest increasing subsequence ending at } i.$

If there is no increasing subsequence of length $p + 1$, then

$$1 \leq a_i \leq p.$$

Now consider the sequence grouped according to the values of a_i . There are at most p possible values.

If the total length is $pq + 1$, then by the pigeonhole principle there exists some value k such that at least $q + 1$ elements satisfy $a_i = k$.

Take those elements in increasing index order. Since they all have the same a_i , an earlier value cannot be smaller than a later value: appending the later value would extend an increasing subsequence and give it a larger label. The values are distinct, so they must form a decreasing sequence.

Thus we obtain a decreasing subsequence of length $q + 1$.

Therefore every sequence of $pq + 1$ distinct real numbers contains either an increasing subsequence of length $p + 1$ or a decreasing subsequence of length $q + 1$.

9. SET 9: PERMUTATIONS AND COMBINATIONS ¹¹

Problem 9.1 (Exercise 79 part (i)). There are $p \geq 6$ distinct points given on the circumference of a circle, and every two points are joined by a chord. Find the number of such chords. For this problem and the next three, assume that no three distinct chords pass through the same point in the interior of the circle. This is the configuration used in the following solutions.

Solution 9.1 (Exercise 79 part (i)). We have a total of $\binom{p}{2}$ chords, since each pair of the p points determines exactly one chord. The concurrency condition is not needed for this first count.

Problem 9.2 (Exercise 79 part (ii)). Find the number of points of intersections of chords within the interior of circle.

Solution 9.2 (Exercise 79 part(ii)). We choose four of the p points on the circle. In their cyclic order, they form a quadrilateral, and its two diagonals have exactly one intersection in the interior. Conversely, every pair of chords intersecting in the interior has four different endpoints, so it arises in this way.

Since no three chords meet at an interior point, different crossing pairs give different intersection points. Thus, the total number of interior intersections is $\binom{p}{4}$.

Problem 9.3 (Exercise 79 part (iii)). Find the number of line segments obtained through dividing the chords by their points of intersection.

Solution 9.3 (Exercise 79 part (iii)). We notice that there are $\binom{p}{2}$ chords. Before dividing them, each chord is one segment. At each interior crossing, exactly two chords meet, and adding that point divides one segment on each of those two chords into two. Thus, each crossing adds two to the total number of segments.

There are $\binom{p}{4}$ crossings by the previous problem. Therefore, the total number of segments is

$$\binom{p}{2} + 2\binom{p}{4}.$$

Problem 9.4 (Exercise 79 part (iv)). Find the number of triangles whose vertices is the intersection of the chords in the interior of the circle.

Distinguish triangles whose sides must lie along the drawn chords, allowing other crossings on their sides, from triangles formed by joining any three noncollinear interior intersection points.

Solution 9.4 (Exercise 79 part (iv)). First, suppose the sides must lie along the drawn chords. Such a triangle uses three chords which cross each other in the interior. They have six different endpoints, since two chords sharing an endpoint do not cross in the interior.

Conversely, choose six points and label them $v_1, \dots, v_6$ in cyclic order. The three chords v_1v_4 , v_2v_5 and v_3v_6 cross each other. They are the only pairing with this property: each chord must have two of the other endpoints on each side, so its endpoints must be opposite in this cyclic list. By the no-three-concurrent condition, their three intersections are distinct and form one triangle. Thus, the count under this interpretation is $\binom{p}{6}$.

Now suppose any three interior intersection points may be joined by new segments. Let I be the set of all interior intersections. We have $|I| = \binom{p}{4}$. We must remove the collinear triples from all triples in I . Thus, the answer is

$$\binom{|I|}{3} - \sum_{\ell} \binom{|I \cap \ell|}{3},$$

where the sum is over all lines ℓ containing at least three points of I . Each collinear triple belongs to just one line, so it is removed exactly once. This count uses the collinearities of the actual

¹¹Principles and Techniques in Combinatorics by Chen Chuan-Chong, et. al Ch1

drawing and is not the same counting problem as the first interpretation. Neither count is restricted to triangular faces.

Problem 9.5 (Exercise 80). Let $n \geq 1$ be an integer. In how many ways can $n + 1$ different prizes be awarded to n students in such a way that each student has at least one prize?

Solution 9.5 (Exercise 80). Given that each student has at least one prize, there are n ways to choose which student gets the additional prize. Since the prizes are distinct, there are $\binom{n+1}{2}$ ways to select which two prizes are given to the student with one additional prize than all the others. We must also consider that there are $n + 1 - 2 = n - 1$ remaining prizes, and $(n - 1)!$ ways to arrange them for the other $n - 1$ students. By multiplication principle, there are $n \cdot \binom{n+1}{2} \cdot (n - 1)!$ ways for each student to be given at least one prize. This simplifies to $n \cdot \frac{(n+1)!}{2!(n+1-2)!} \cdot (n - 1)!$. Thus, there are $\frac{n(n+1)!}{2}$ ways to award the different prizes.

Problem 9.6 (Exercise 81 part (a)). Let $n, m, k \in \mathbb{N}$ and let $N_k = \{1, 2, \dots, k\}$. Find the number of mappings from N_n to N_m and the number of one-to-one mappings.

Solution 9.6 (Exercise 81 part(a)). We know that each of the n elements in N_n has m choices for its output in N_m . Thus, the total number of mappings is m^n .

For a mapping to be one-to-one, different inputs must have different outputs. If $n \leq m$, we have m choices for the first output, $m - 1$ for the second, and so on, giving

$$m(m - 1) \cdots (m - n + 1) = \frac{m!}{(m - n)!}.$$

If $n > m$, there are no one-to-one mappings, by the pigeonhole principle. The answer $n!$ occurs only in the special case $n = m$.

Problem 9.7 (Exercise 81 part(b)). A mapping of $f : N_n \rightarrow N_m$ is strictly increasing if $f(a) < f(b)$ wherever $a < b$ in N_n . Find the number of strictly increasing mappings from N_n to N_m where $n \leq m$.

Solution 9.7 (Exercise 81 part(b)). We realize that for any set of n elements in N_m there is only one satisfactory permutation fulfilling the *increasing* condition. Thus, we are basically finding the number of n -element combinations in N_m , which is $\binom{m}{n}$.

For an image set $\{b_1 < \cdots < b_n\}$, the map must be $f(i) = b_i$. Conversely, this rule gives a strictly increasing map for every such image set.

Problem 9.8 (Exercise 88). Let $n \in \mathbb{N}$. How many solutions are there in ordered positive integer pairs (x, y) to the equation $\frac{xy}{x+y} = n$?

Solution 9.8. Multiply by $x+y$ on both sides to get

$$(3) \quad xy = nx + ny$$

Then subtract the RHS from both sides to get

$$(4) \quad xy - nx - ny = 0$$

You can then add n^2 and factor to get

$$(5) \quad (x - n)(y - n) = n^2$$

Note that $xy/(x+y) < \min(x, y)$ for positive x, y , so any original solution has $x > n$ and $y > n$. Thus, $x - n$ is a positive divisor d of n^2 , and

$$x = n + d, \quad y = n + \frac{n^2}{d}.$$

Conversely, every positive divisor d gives positive integers by this rule. They satisfy $(x - n)(y - n) = n^2$, which expands to $xy = n(x + y)$ and hence to the original equation. Different divisors give different first coordinates. Therefore, the number of ordered solutions is the number of positive divisors of n^2 .

Problem 9.9 (Exercise 89 part (i)). Let $S = \{1, 2, \dots, 1992\}$. In each of the following cases find the number of 3-elements subsets $\{a, b, c\}$ such that $3|a + b + c$

Solution 9.9 (Exercise 89 part (i)). We notice that there are $\frac{1992}{3} = 664$ numbers for each residue of 3 (0,1,2). We also notice that there are 2 possible cases for a,b,c in terms of mod 3 such that $a + b + c \equiv 0 \pmod{3}$.

Case 1: a,b and c are all same mod 3

In this case, we have 3 options to choose which residue class modulo 3. We then have $664 \cdot 663 \cdot 662$ total permutations. Thus, we have a total of $\frac{3 \cdot 664 \cdot 663 \cdot 662}{6} = 3 \binom{664}{3}$ sets.

Case 2: a,b, and c are all different mod 3

WLOG, a is 0 mod 3, b is 1 mod 3 and c is 2 mod 3. Thus, we have a total of 664^3 sets.

In total, we have $3 \binom{664}{3} + 664^3$ sets a,b,c fulfilling the condition that $3|a + b + c$

Problem 9.10 (Exercise 89 part (ii)). Let $S = \{1, 2, \dots, 1992\}$. In each of the following cases find the number of 3-elements subsets $\{a, b, c\}$ such that $4|a + b + c$

Solution 9.10 (Exercise 89 part (ii)). We notice that 1992 is divisible by 4. To show that the sums of the three-element subsets are equally distributed modulo 4, shift each element one place in the cycle

$$1 \mapsto 2 \mapsto \dots \mapsto 1992 \mapsto 1.$$

This is a permutation of S, and every element increases by 1 modulo 4, including 1992 when it changes to 1. Thus, applying the shift to a three-element subset increases its sum by 3 modulo 4.

The shift is invertible, so it gives a bijection between the classes of subsets whose sums differ by 3 modulo 4. Repeating it connects all four residues. Therefore, these four classes have equal sizes, and the answer is

$$\frac{1}{4} \binom{1992}{3} = 328,853,470.$$

Problem 9.11 (Exercise 90). A sequence of 15 independent uniform random draws, one at a time with replacement is made from the set A,B,C... X,Y,Z. What is the probability that UNIVERSITY occurs as a block in the sequence.

Solution 9.11 (Exercise 90). We notice that there is a total of 26^{15} total possible permutations. Out of those, if we consider UNIVERSITY to be a block, we still have 5 other letters to fill out. The block can start at positions 1 through 6. Two occurrences cannot happen in the same sequence: they would overlap, putting a second U inside the first occurrence, but UNIVERSITY has no U after its first letter. Thus, we have a total of $6 \cdot 26^5$ combinations with the block UNIVERSITY in the sequence. Thus, the probability is $\frac{6 \cdot 26^5}{26^{15}} = \frac{6}{26^{10}}$.

Problem 9.12 (Exercise 94). A gardener plants three maple trees, four oak trees, and five birch trees in a row. He plants them in random order, each arrangement being equally likely. Let $\frac{m}{n}$ in lowest terms be the probability that no two birch trees are next to one another. Find $m + n$.

Solution 9.12 (Exercise 94). The total number of arrangements of the trees in random order is $\frac{12!}{3! \cdot 5! \cdot 4!}$. Now we must consider how many ways there are to arrange the three maple trees and the

four oak trees, then insert the birch trees such that no two are consecutive. There are $\frac{7!}{3!4!}$ ways to arrange the maple and oak trees. We notice that there are eight gaps to place birch trees to ensure that no two are consecutive. Thus, there are $\binom{8}{5}$ ways to arrange the birch trees. Since the birch trees are indistinguishable objects, we don't need to multiply by any further factor. Therefore, the probability that no two birch trees are next to one another can be represented as $\frac{\binom{8}{5} \cdot \frac{7!}{3!4!}}{\frac{12!}{3!4!5!}}$. This simplifies to $\left(\frac{8!}{5!3!} \cdot \frac{7!}{4!3!}\right) \cdot \frac{3!4!5!}{12!}$, which can be further reduced to $\frac{7 \cdot 6 \cdot 5 \cdot 4}{12 \cdot 11 \cdot 10 \cdot 9} = \frac{7}{99}$. Thus, $m + n = 106$.

Problem 9.13 (Exercise 95). In a tournament, each player played exactly one game against all the other players. In each game, the winner was awarded 1 point, the loser got 0 points and two players earned $\frac{1}{2}$ points or a tie. After the completion of a tournament, it was found that exactly half the points earned by each player were earned in games against 10 players with the least number of points. In particular, each of the ten lowest scoring players earned half their points against the other 9 players. How many players were in that tournament?

Solution 9.13 (Exercise 95). Assuming that there were $x+10$ players in the tournament, we would have a total of 45 games with a possible 45 between the 10 weakest players. Also the x players played a total of $\binom{x}{2}$ games among each other.

Now, we notice that the total number of points scored by the 10 players out of those 45 games would be 45. This is also half the total number of points they earned.

We also notice that the x players earned a total of $\binom{x}{2}$ points among each other and this too would only be half the number of points they would earn in total.

This means that that $45 + \binom{x}{2}$ would represent only *half* the total number of points earned. However, the total number of points earned would also be $\binom{x+10}{2}$.

Thus, $2(45 + \binom{x}{2}) = \binom{x+10}{2} \rightarrow 90 + x(x-1) = \frac{(x+10)(x+9)}{2} \rightarrow x^2 - 21x + 90 = 0$.

This means that $x=6$, $x=15$ which means $n=16$ and $n=25$ and we have two potential solutions for the total number of players. Thus, we realize that the number of games the x players played against the bottom 10 players would be $10x$.

If we have 16 players however, then the top 6 would earn a total of $15 \cdot 2 = 30$ points. The bottom 10 would earn $45 \cdot 2 = 90$ points. However, we arrive at a contradiction because this means that the bottom 10 averaged $90/10 = 9$ points while the top 6 averaged $30/6 = 5$ points, contrary to the ordering of the two groups. Thus, the answer is 25 people.

Problem 9.14 (Exercise 96). Let S be the sum of base ten logarithms of all of the proper divisors of 1,000,000. (By a proper divisor of a natural number we mean a positive integral divisor other than 1 and the number itself). What is the integer nearest to S ?

Solution 9.14 (Exercise 96). We notice that 1,000,000 is equal to $2^6 \cdot 5^6$. The sum of logarithm of all divisors of 1,000,000 would just be the logarithm of the product of all divisors by using log properties. Thus, if M is the product of all positive divisors of 1000000, then the sum of their base ten logarithms is $\log_{10} M$.

The first step is to realize that every power of 2 repeats 7 times, one for every single power of 5. The same thing goes for every power of 5. Thus, the exponent for 2 is $7(0+1+2+3+4+5+6) =$

147 and the same goes for 5. This means $M = 10^{147}$. However, we exclude the $2^6 \cdot 5^6$ since that is not a proper divisor and the product of the proper divisors becomes 10^{141} . Excluding the divisor 1 does not change the product. Thus, $S = \log_{10}(10^{141}) = 141$, and our answer is 141.

Problem 9.15 (Exercise 98). An ordered pair (m, n) of non-negative integers is called 'simple' if the addition $m+n$ in base 10 requires no carrying. Find the number of simple ordered pairs of non-negative integers that sum up to (i) 1492 (ii) 1992.

Solution 9.15 (Exercise 98). For a target digit d , the ordered pair of addend digits can be $(0, d), (1, d-1), \dots, (d, 0)$, giving $d+1$ choices. Leading zeros are allowed in the addends, and no carrying makes the choices at different positions independent. We call the two numbers as number 1 and number 2. We have 3 options for the unit digit. We then have 10 options for the tens digit, 5 options for the hundreds digit and 2 options for the thousands digit, giving us a total of $3 \cdot 10 \cdot 5 \cdot 2 = 300$ pairs of integers.

Similarly, for the 1992 numbers, we have $3 \cdot 10 \cdot 10 \cdot 2 = 600$ pairs of integers.

Problem 9.16 (Exercise 99). Let $\frac{m}{n}$, in the lowest terms, be the probability that a uniformly chosen positive divisor of 10^{99} is an integer multiple of 10^{88} . Find $m+n$.

Solution 9.16 (Exercise 99). We notice that 10^{99} has $(100)(100)$ factors. Of those factors, the only ones which are a multiple of 10^{88} would be those that have a higher exponent than 87 for both 2 and 5. This would give us a total of $12(12)$ factors which are an integer multiple of 10^{88} . Thus, the probability of selecting one of these factors is $\frac{12 \cdot 12}{100 \cdot 100} = \frac{9}{625}$. Thus, $m+n = 634$.

10. SET 10: PERMUTATIONS AND COMBINATIONS¹²

Problem 10.1 (Exercise 1). How many 3-digit positive integers are there, such that three distinct digits appear in the decimal representation of any of them?

Solution 10.1 (Exercise 1). Start by considering the number of ways to choose the first digit of the 3-digit positive integer. There are 9 choices, the numbers 1-9. Now consider the number of ways to choose the second digit. There are also 9 choices, 0 and the remaining numbers from 1-9 to exclude repetition. Finally, we notice that there are $10-2 = 8$ choices for the third digit in the number. Therefore, by the multiplication principle, there are $9 \cdot 9 \cdot 8 = 648$ 3-digit positive integers with three distinct digits.

Problem 10.2 (Exercise 2). How many 5-digits positive integers are there that are divisible by 5 and without a repetition of digits?

Solution 10.2 (Exercise 2). For numbers to be divisible by five, there are two cases: either the last digit of the number is 5 or 0.

Case 1: Last digit is 5 We can choose the first digit in 8 ways (the digits 1-9 excluding 5). For the second digit, we may now include 0 in the count. Thus, the second digit can be selected in 8 ways. The third digit has 7 possibilities and the fourth digit has 6 possibilities. By multiplication principle, this totals to $8 \cdot 8 \cdot 7 \cdot 6 = 2688$ valid numbers.

Case 2: Last digit is 0 We can choose the first digit in 9 ways (the digits 1-9). The second digit has 8 potential candidates, the third digit has 7 and the fourth digit has 6. By multiplication principle, this totals to $9 \cdot 8 \cdot 7 \cdot 6 = 3024$ valid numbers.

Therefore, by the addition principle, there are $2688 + 3024 = 5712$ 5-digit positive integers that are divisible by 5 without a repetition of digits.

Problem 10.3 (Exercise 3). Twelve basketball teams take part in a tournament. How many possible ways are there for the distribution of gold, silver, and bronze medals?
The medals go to three different teams, with no ties.

Solution 10.3 (Exercise 3). Given that there are 12 teams, we must choose 3 to win medals. Since the order of the medal winners matters, we must multiply by the $3! = 6$ internal permutations of gold, silver, and bronze medals. Therefore, there are $\binom{12}{3} \cdot 3! = 1320$ ways to distribute the medals.

Problem 10.4 (Exercise 4). There are n light bulbs in a room, each of them with its own switch. How many ways can the room be lighted?

Solution 10.4 (Exercise 4). Each light bulb can either be on or off. Thus, there are 2^n ways the room can be lighted.

This counts all switch configurations, including the one with every bulb off. If the room must have at least one bulb on, we exclude that one configuration and obtain $2^n - 1$.

Problem 10.5 (Exercise 5 (a)). How many divisors of natural number 2000 are there?

Solution 10.5 (Exercise 5 (a)). The prime factorization of 2000 is $2^4 \cdot 5^3$. Hence, there are 5 choices for the exponent of 2, from 0 through 4, and four choices for the exponent of 5, from 0 through 3. Therefore, there are $4 \cdot 5 = 20$ natural divisors of 2000.

¹²Combinatorics: A Problem-Based Approach by Pavle Mladenović

Problem 10.6 (Exercise 5 (b)). Let $p_1, p_2, \dots, p_m$ be distinct prime numbers, and $k_1, k_2, \dots, k_m$ be positive integers. Determine the number of divisors of the positive integer $n = p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$.

Solution 10.6 (Exercise 5 (b)). By the previous part, we can deduce that the general formula for the number of divisors of the positive integer is $(k_1 + 1)(k_2 + 1) \dots (k_m + 1)$. This works since for each distinct prime number divisor, you can choose the exponent to be: 0, 1, 2, ..., k_i . Every divisor has exactly one such choice of exponents by unique prime factorization, and no other primes or larger exponents are allowed.

Problem 10.7 (Exercise 6). Determine all positive integers with an odd number of divisors.

Solution 10.7 (Exercise 6). For $(k_1 + 1)(k_2 + 1) \dots (k_m + 1)$ to be odd, all k_i must be even because every $k_i + 1$ factor must be odd. Thus, every k_i can be written as $2 \cdot$ (an integer). This means that $n = p_1^{2i_1} p_2^{2i_2} \dots p_m^{2i_m} = (p_1^{i_1} p_2^{i_2} \dots p_m^{i_m})^2$. Thus, all positive integers with an odd number of divisors are perfect squares.

Conversely, if $n > 1$ is a perfect square, every prime exponent is even. Each factor $k_i + 1$ is then odd, so their product, the number of divisors, is odd. The number 1 also belongs to the answer: it is a perfect square and has exactly one positive divisor.

Problem 10.8 (Exercise 7). In a big country no two citizens have the same arrangement of teeth. What is the maximal possible number of citizens in this country?

In this model there are 32 specified tooth positions, each either occupied or unoccupied, and only this information distinguishes arrangements.

Solution 10.8 (Exercise 7). Given the 32 specified positions in the model, each position has two choices: the tooth is present or absent. Thus, there are $2^{32} = 4,294,967,296$ arrangements. Since no two citizens can have the same arrangement, this is the maximum number in the model, attained by using every possible arrangement once.

Problem 10.9 (Exercise 8). Let $n \geq 2$. How many permutations of the set $\{1, 2, \dots, n\}$ are there, such that 1 and 2 are adjacent terms?

Solution 10.9 (Exercise 8). Consider the numbers 1 and 2 as a group. There are $(n - 1)!$ ways to arrange this group and the other $n - 2$ elements of the set $\{1, 2, \dots, n\}$. However, we must also consider the internal permutations of the group. There are $2! = 2$ ways to arrange the elements of the group. Therefore, there are $2(n - 1)!$ permutations of the set, such that 1 and 2 are adjacent terms.

Problem 10.10 (Exercise 9). Let $n \geq 2$. How many permutations of the set $\{1, 2, \dots, n\}$ are there, such that 2 is placed after 1, not necessarily at adjacent positions?

Solution 10.10 (Exercise 9). For every permutation where 1 is before 2, there is a corresponding permutation where 2 is before 1, obtained by swapping the labels 1 and 2. Swapping again recovers the original permutation, so this pairing is a bijection. Thus, exactly half of the total permutations have 1 placed before 2, and vice versa. Therefore, there are $\frac{n!}{2}$ permutations of the set $\{1, 2, \dots, n\}$, such that 2 is placed after 1.

Problem 10.11 (Exercise 10). Let n, k be integers with $0 \leq k \leq n - 2$. How many permutations of the set $\{1, 2, \dots, n\}$ are there, such that exactly k elements appear between 1 and 2.

Solution 10.11 (Exercise 10). We can consider the arrangement to be as one 'block' of 1, 2 and the k elements in between and the rest of the $n - k - 2$ elements. Thus, we have a total of $(n - k - 1)!$ ways to arrange the total number of elements left and the 'block'. We also have $2 \binom{n-2}{k} k!$ ways to arrange the 'block'. Thus, we have a total of $2(n - k - 1)! \binom{n-2}{k} k! = 2(n - k - 1)!(n - 2)!$.

Problem 10.12 (Exercise 11). Let $n \geq 1$ be an integer. How many permutations $(a_1, a_2, \dots, a_{3n})$ of the set $\{1, 2, \dots, 3n\}$ are there, such that, for any $k \in \{1, 2, \dots, 3n\}$, the difference $k - a_k$ is divisible by 3?

Solution 10.12 (Exercise 11). To ensure that the difference $k - a_k$ is divisible by 3 for every permutation, we can partition the set into the disjoint subsets. The first set contains numbers with remainder 1 mod 3: $\{1, 4, \dots, 3n - 2\}$. The second set includes numbers with remainder 2 mod 3: $\{2, 5, \dots, 3n - 1\}$. The third set contains numbers with no remainder mod 3: $\{3, 6, \dots, 3n\}$. Within each of these sets, there are n numbers, and these numbers can be permuted among the n positions of the same residue in $n!$ ways. Thus, there are $(n!)^3$ permutations of the set $\{1, 2, \dots, 3n\}$, such that $k - a_k$ is divisible by 3.

Problem 10.13 (Exercise 12). How many ways can eight white chess pieces (the king, the queen, two rooks, two knights, and two bishops) be arranged on the first row of a chessboard?

Solution 10.13 (Exercise 12). There are $8!$ ways to arrange eight distinguishable chess pieces on the board. However, we must consider the indistinguishable objects: two rooks, two knights, and two bishops. Thus, there are $\frac{8!}{2! \cdot 2! \cdot 2!} = 5040$ ways to arrange the eight white chess pieces on the first row of a chessboard.

Problem 10.14 (Exercise 13). How many ways can n people sit around a circular table, if the chairs are not labeled?

Take $n \geq 1$ distinct people, with rotations regarded as the same seating and reflections not identified.

Solution 10.14. When arranging objects in a circle, we must consider how only the relative positions of the objects matter because rotations are considered the same permutation. Since every arrangement of people in a circle would be over counted n times in the $n!$ linear permutations. We must divide by n to remove each identical permutation of rotations. Therefore, there are $\frac{n!}{n} = (n - 1)!$ ways to arrange people around a circular table.

Problem 10.15 (Exercise 14). Given $n \geq 3$ points in a plane such that no three of them belong to the same line, how many lines are determined by these points.

Solution 10.15 (Exercise 14). Since no three points are on the same line, that means that the total number of lines is simply $\binom{n}{2}$.

Every pair determines a line, and two different pairs cannot give the same line without at least three of the given points lying on that line.

Problem 10.16 (Exercise 15). Given $n \geq 3$ points in space, such that no three are collinear and no four belong to the same plane, how many planes are determined by these points.

Solution 10.16 (Exercise 15). Using a similar reasoning as in [Solution 10.15](#), every three of the points determine exactly one plane because they are not collinear. Two different triples cannot determine the same plane, since their union would contain at least four coplanar points. Thus, the total number of planes is $\binom{n}{3}$.

Problem 10.17 (Exercise 17). A group consists of six mathematicians. How many ways can they form five committees each consisting of three members, such that no two of those committees consist of the same three mathematicians?

The five committees form an unordered collection; they do not have different names or offices.

Solution 10.17 (Exercise 17). There are a total of $\binom{6}{3} = 20$ groups of mathematicians and $\binom{20}{5} = 15504$ total ways to make the committees.

Problem 10.18 (Exercise 18). Let us take a deck of cards consisting of 8 red, 8 blue, 8 yellow, and 8 green cards. Cards of the same color are numbered 1, 2, ..., 8. How many ways can 6 cards be chosen such that at least one card of each color is chosen?

Solution 10.18 (Exercise 18). We must divide this problem into cases that distribute the 6 cards into 4 colors with at least one in each.

Case 1 - (1, 1, 1, 3): This happens in $\binom{8}{1}^3 \cdot \binom{8}{3}$ ways. We must multiply by the $\frac{4!}{3!} = 4$ permutations of this case. Thus, there are $4 \cdot 8^3 \cdot \binom{8}{3}$ ways to choose the cards with this color distribution.

Case 2 - (2, 2, 1, 1): This occurs in $\binom{8}{2}^2 \cdot \binom{8}{1}^2$ ways. Now we multiply by the $\frac{4!}{2!2!} = 6$ permutations of this case. Hence, there are $6 \cdot 8^2 \cdot \binom{8}{2}^2$ ways to choose the cards with this color distribution.

By the addition principle, there are $4 \cdot 8^3 \cdot \binom{8}{3} + 6 \cdot 8^2 \cdot \binom{8}{2}^2 = 415\,744$ ways the 6 cards be chosen such that at least one card of each color is chosen.

Problem 10.19 (Exercise 19). Peter has 12 relatives (five men and seven women), and his wife also has 12 relatives (five women and seven men). They do not have common relatives. They decide to invite 12 guests, six each of their relatives, such that there are six men and six women among the guests. How many ways can they choose 12 guests?

Solution 10.19 (Exercise 19). Let r represent the number of male relatives we choose from Peter's side of the family. This can be done in $\binom{5}{r}$ ways. Now we can choose the female relatives from Peter's side of the family in $\binom{7}{6-r}$ ways. There are also $\binom{7}{6-r}$ ways to choose the male relatives from Peter's wife's side of the family. Lastly, we can choose the female relatives from Peter's wife's side of the family in $\binom{5}{r}$ ways. Since $6-r + 6-r + r + r = 12$ guests, we can verify that this works. By the multiplication principle, this can be expressed as $[\binom{7}{6-r} \cdot \binom{5}{r}]^2$. Now we must sum over all valid r , from 0 to 5. Therefore, they can choose 12 guests: six each of their relatives, six men and six women in $\sum_{r=0}^5 [\binom{7}{6-r} \cdot \binom{5}{r}]^2 = 267\,148$ ways.

Problem 10.20 (Exercise 20). Let n, k be nonnegative integers with $k \leq n + 1$. How many ways can n 0's and k 1's be arranged in a sequence, such that no two 1's are adjacent?

Solution 10.20 (Exercise 20). First arrange the zeros. Since the zeros are identical, permutations don't change anything and there is only one way to arrange the zeros in a sequence. Now insert the ones in the gaps between the zeros, including the endpoints. There are $n + 1$ such gaps that guarantee no two 1's are adjacent. Hence, we must choose k of these $n + 1$ gaps to place the ones in. Therefore, there are $\binom{n+1}{k}$ ways to form the desired sequences.

Problem 10.21 (Exercise 21). Twelve books are arranged in a sequence. How many ways can five books be chosen such that no two of them are adjacent in the sequence?

Solution 10.21 (Exercise 21). We mark each chosen position by 1 and each unchosen position by 0. Thus, we need a sequence with five 1s and seven 0s, with no two 1s adjacent. Fix the seven 0s, which create eight gaps, including the ends. We choose five of these eight gaps for the 1s. This can be done in $\binom{8}{5} = 56$ ways. Each resulting pattern selects exactly one set of positions in the original line of books, so no rearrangement of the books is being counted.

Problem 10.22 (Exercise 22). Twelve knights are arranged around King's Arthur Round Table. All of them are quarreling with their neighbors. How many ways can King Arthur choose five

knights such that no two of them are quarreling?

Assume that only neighbouring knights are quarreling.

Solution 10.22 (Exercise 22). This problem would be identical to the previous, if King Arthur's Round Table were not circular. The challenge lies in the fact that seat 1 and seat 12 are in fact adjacent and would be quarreling with each other. Thus, we must divide the solution into two disjoint linear cases.

Case 1 - Seat 1 is chosen: Given that seat 1 is chosen, neither seat 2 nor seat 12 can be chosen. The nine remaining seats are numbered 3-11. We may now insert the gaps in the same way as the previous problem to avoid adjacency. There are $9 - 4 = 5$ unchosen seats, which create 6 gaps (including ends). Thus, there are $\binom{6}{4}$ ways to place the chosen knights.

Case 2 - Seat 1 is not chosen: With the first seat not chosen, there are no restrictions on seat 2 or seat 12. Hence, there are 11 remaining seats to place the five selected knights. The six unchosen seats leave seven gaps to place to knights. Hence, there are $\binom{7}{5}$ ways to place the chosen knights, such that no two of them are quarreling.

By the addition principle, there are $\binom{6}{4} + \binom{7}{5} = 15 + 21 = 36$ ways King Arthur can choose the five knights from his Round Table.

Problem 10.23 (Exercise 23). Let $n \geq 1$ be an integer. How many n -digit positive integers $c_1c_2\dots c_n$ are there, such that $1 \leq c_1 \leq c_2 \leq \dots \leq c_n \leq 9$?

Solution 10.23 (Exercise 23). For this problem, we can apply a stars and bars approach because we are dividing indistinguishable objects (how many times each digit appears) into boxes according to the value of each digit. Since there are 9 possible digits and the sequence must be non-decreasing, there are nine boxes for the numbers: (digit one), (digit two), (digit three) ... (digit nine). Therefore, by stars and bars, there are $\binom{n+9-1}{9-1} = \binom{n+8}{8}$ n -digit positive integers that satisfy the criteria.

Conversely, once the multiplicity of every digit is chosen, writing the digits in increasing order gives exactly one allowed number. Thus, no additional permutation factor is needed.

Problem 10.24 (Exercise 24). How many 6-digit positive integers are there such that there are exactly three distinct digits in the decimal representation of each of them, and the digit 0 does not appear?

Solution 10.24 (Exercise 24). There are $\binom{9}{3}$ ways to choose which three digits appear in the 6-digit positive integer. Now we must think of the different distributions for the occurrence of numbers in a 6-digit positive integer with exactly three distinct digits.

Case 1 - (1, 1, 4): Among the three chosen numbers, we must choose which digit appears four times. This can be done in $\frac{3!}{2!} = 3$ ways. The number of arrangements is $\frac{6!}{4! \cdot 1! \cdot 1!}$ ways.

Case 2 - (1, 2, 3): We must choose which digit occurs once, twice and three times. This gives $3! = 6$ ways. The number of arrangements is $\frac{6!}{1! \cdot 2! \cdot 3!}$ ways.

Case 3 - (2, 2, 2): For each digit to appear twice, there is only $\frac{3!}{3!} = 1$ way to assign the frequencies. The number of arrangements is $\frac{6!}{2! \cdot 2! \cdot 2!}$.

Therefore, there are $\binom{9}{3} \cdot (3 \cdot \frac{6!}{4! \cdot 1! \cdot 1!} + 6 \cdot \frac{6!}{1! \cdot 2! \cdot 3!} + \frac{6!}{2! \cdot 2! \cdot 2!}) = 84(90 + 360 + 90) = 45360$ ways.

Problem 10.25 (Exercise 25). How many different words can be obtained by permuting the letters of the word locomotive such that no two O s are adjacent.

Solution 10.25 (Exercise 25). The seven letters other than O are distinct. We use O_1, O_2, O_3 only to name the first, second and third O in left-to-right order; the O s themselves remain indistinguishable. We think of the permutations as $3O$ s and we need to fill out the other letters in between the gaps. Now, we have a total of 4 ways to put the L , 5 ways to put the letter C and so on, yielding a total of $4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 = \frac{10!}{3!}$.

Now, we examine all possible cases where O s are adjacent to each other. This happens in two different ways:

- (1) The gap between O_1 and O_2 is empty
- (2) The gap between O_2 and O_3 is empty

Now, we see that Case 1 is just 1 less gap for the letters to go through, making it equal to $\frac{9!}{2!}$ and the same holds for Case 2. Thus, the total sum is just $9!$. However, we over-count when both Case 1 and Case 2 hold true. That is a total of $8!$ ways.

Therefore, the total number of ways is $\frac{10!}{3!} - 9! + 8! = 282240$ ways.

Problem 10.26 (Exercise 26). Let $n \geq 1$ and $k \geq 0$ be integers. All k -arrangements of elements $a_1, a_2, \dots, a_n$ are written down. Determine how the many times the element a_1 is written down. Give the count both when repetitions are allowed and when they are not allowed.

Solution 10.26 (Exercise 26). First suppose repetitions are allowed and $n \geq 1, k \geq 1$. The total number of appearances of all elements in the n^k words is kn^k . Each element occurs equally often, so the number of appearances of a_1 is kn^{k-1} . We can also see this by choosing the position of a counted occurrence in k ways and filling the other positions in n^{k-1} ways. This counts separate occurrences even when one word contains a_1 several times.

Now suppose repetitions are not allowed and $1 \leq k \leq n$. We choose the position of a_1 in k ways and fill the other $k - 1$ positions with different elements from the remaining $n - 1$. This gives

$$k \frac{(n-1)!}{(n-k)!}$$

appearances. If $k > n$ there are no such arrangements without repetition. For $k = 0$, the empty arrangement has no appearances, so the count is zero in either convention.

Problem 10.27 (Problem 27). A box contains 36 yellow, 27 blue, 18 green, and 9 red balls. Balls of the same color are indistinguishable. How many ways are there to choose 10 balls from the box?

Solution 10.27 (Problem 27). We write the four color counts as $y, b, g, r \geq 0$, with $y+b+g+r = 10$. Using stars-and-bars we get a total of $\binom{13}{3} = 286$ possible ways. However, one of the ways is 10 red balls which can't happen since there are only 9 red balls so it is just 285 ways.

Problem 10.28 (Exercise 28). Determine the number of 0 s at the end of the decimal representation of the positive integer $144!$.

Solution 10.28 (Exercise 28). We know that the number of 0 s is influenced by how many times 10 can divide $144!$. Since $10 = 2 \cdot 5$, and 5 is less often than 2, which means that we need to find the number of times 5 is in the prime factorization of $144!$. This is equal to

$\lfloor \frac{144}{5} \rfloor + \lfloor \frac{144}{25} \rfloor + \lfloor \frac{144}{125} \rfloor = 28 + 5 + 1 = 34$. Thus, there are 34 zeroes at the end of 144!

The multiples of 25 and 125 are counted again because they contribute additional factors of 5.

Problem 10.29 (Exercise 29). Prove that the product of k consecutive positive integers is divisible by $k!$.

Solution 10.29 (Exercise 29). Let $n, k \geq 1$ be integers. The product of consecutive numbers can be expressed as: $n(n+1)(n+2)\dots(n+k-1)$. As factorials, this can be rewritten in the form of $\frac{(n+k-1)!}{(n-1)!}$ to represent $\frac{(n+k-1)(n+k-2)(n+k-3)\dots n\dots 2\cdot 1}{(n-1)!}$. Thus, we are trying to prove that $\frac{(n+k-1)!}{k!(n-1)!}$ results in an integer. This fraction is the binomial coefficient $\binom{n+k-1}{k}$, which counts k -element subsets of an $(n+k-1)$ -element set. Since $\binom{n+k-1}{k}$ always results in an integer, the proof is complete.

Problem 10.30 (Exercise 30). Let $a_1, a_2, \dots, a_n$ be nonnegative integers. Prove that

$$a_1!a_2!\dots a_n! \leq (a_1 + a_2 + \dots + a_n)!.$$

Solution 10.30 (Exercise 30). We can prove that the RHS is greater than or equal to the LHS by considering the number of permutations of a word of length $a_1 + a_2 + \dots + a_n$ with repeated letters. This would be expressed as $\frac{(a_1+a_2+\dots+a_n)!}{(a_1!a_2!\dots a_n!)}$, where the denominator accounts for the repeated letters. Since this fraction is always a positive integer, it is evident that $a_1!a_2!\dots a_n! \leq (a_1 + a_2 + \dots + a_n)!$.

Problem 10.31 (Exercise 31). Determine the sum of all 4-digit positive integers, such that there are two different even digits and two equal odd digits in the decimal representation of each of them.

Solution 10.31 (Exercise 31). We let the four digit number be $\overline{ABCD}$. We break it down into 2 cases: if A is odd or if A is even. In each case, expected value means the average over all allowed numbers in that case, with each such number given the same probability.

Case 1: A is odd

There are 5 choices for the odd digit, 3 choices for the other position carrying that digit, and $5 \cdot 4$ ordered choices for the two different even digits. Thus, there are $5 \cdot 3 \cdot 5 \cdot 4 = 300$ numbers. The expected value of A is 5. The remaining positions contain one odd digit of mean 5 and two even digits each of mean 4. Their positions are symmetric, so the average digit in each remaining position is $(5 + 4 + 4)/3 = 13/3$. Hence, the mean number is

$$5000 + (100 + 10 + 1) \frac{13}{3} = 5481.$$

The sum in this case is $5481 \cdot 300 = 1,644,300$.

Case 2: A is even

A has 4 nonzero even choices. There are 4 choices for the other even digit, 5 choices for the repeated odd digit, and 3 choices for the position of the other even digit. Thus, there are $4 \cdot 4 \cdot 5 \cdot 3 = 240$ numbers. We have $E(A) = 5$. Since all five even digits sum to 20, the other even digit has mean $(20 - 5)/4 = 15/4$. The two odd digits each have mean 5. Therefore, the mean number is

$$5000 + \frac{100 + 10 + 1}{3} \left(5 + 5 + \frac{15}{4} \right) = 5508.75.$$

The sum in this case is $5508.75 \cdot 240 = 1,322,100$.

Therefore, the total sum of all numbers is $1,644,300 + 1,322,100 = 2,966,400$.

Problem 10.32 (Exercise 32). All digits 0, 1, ..., 9 are written down on a sheet of paper. After a 180° rotation, the digits 0, 1, and 8 do not change meaning, the digit 6 becomes 9, and 9

becomes 6. The remaining digits lose their meaning. How many 7-digit positive integers are there, such that they remain the same after the 180° rotation?

Solution 10.32 (Exercise 32). When the 7-digit number gets rotated, d_1 swaps positions with d_7 , d_2 with d_6 , d_3 with d_5 and d_4 remains in the same spot. Each pair of digits can be treated as one unit because the numbers possibilities for one digit determines the other. Thus, the only number candidates for d_4 are $\{0, 1, 8\}$. Since d_1 cannot be 0, neither can d_7 . The number candidates for d_1 and d_7 are $\{1, 6, 8, 9\}$. Finally, the other pairs can be any rotatable number, leaving $\{0, 1, 6, 8, 9\}$ as valid candidates. Therefore, by the multiplication principle, there are $4 \cdot 5^2 \cdot 3 = 300$ 7-digit positive integers that remain the same after the 180° rotation.

Problem 10.33 (Exercise 33). Let $n \geq 1$ be an integer. How many 3-combinations $\{x_1, x_2, x_3\}$ of elements $1, 2, 3, \dots, 3n$ are there such that $x_1 + x_2 + x_3$ is divisible by 3.

Solution 10.33 (Exercise 33). We know that there are n elements in each residue class of the modular relation. Either x_1, x_2, x_3 belong to the same residue class or they all belong to different residue classes. Thus, the total number of combinations is $3\binom{n}{3} + n^3$.

Problem 10.34 (Exercise 34). Determine the total number of 0's in the decimal representations of the positive integers $1, 2, \dots, 10^9$.

Solution 10.34. We must solve this problem by focusing on one length of number at a time. Let d denote the length of the d -digit number. Notice that there are $9 \cdot 10^{d-1}$ numbers of length d . For a uniformly chosen d -digit positive integer, using expected value, the first digit of the d -digit number can never be a zero, but each of the remaining $d - 1$ positions has probability $\frac{1}{10}$ of being zero. Thus, the expected number of zeros is $\frac{d-1}{10}$. Thus, there are $9 \cdot 10^{d-1} \cdot \frac{d-1}{10}$ total zeroes in the numbers of length d . One-digit positive integers contribute zero. Summing the other lengths gives $\sum_{d=2}^9 9(d-1) \cdot 10^{d-2} = 9 + 180 + 2700 + 36000 + 450,000 + 5,400,000 + 63,000,000 + 720,000,000 = 788,888,889$. However, we are excluding the positive integer 10^9 in this evaluation as it is denoted by $d = 10$. Hence, we must add the nine zeroes corresponding to the decimal representation of 10^9 . Therefore, there are total $788,888,889 + 9 = 788,888,898$ total 0's in the decimal representations of the positive integers $1, 2, \dots, 10^9$.

Problem 10.35 (Exercise 35). How many positive integers are there in the set

$$S = \{1, 2, \dots, 10^7\},$$

such that the digit 1 does not appear in their decimal representations?

Solution 10.35 (Exercise 35). We can represent the integers from 0 to 9,999,999 as seven-digit strings from 0000000 to 9999999, allowing leading zeros. Each of the seven positions has 9 choices, namely $\{0, 2, 3, \dots, 9\}$, if the digit 1 is not allowed. Thus, there are 9^7 such strings. This count includes 0000000, so we subtract it because the required integers are positive. The remaining endpoint 10^7 contains the digit 1 and contributes nothing. Therefore, there are $9^7 - 1 = 4,782,968$ positive integers in the given set such that the digit 1 does not appear.

Problem 10.36 (Exercise 36). Determine the sum of all digits that appear in the decimal representation of all positive integers from 1 to 1,000,000.

Solution 10.36 (Exercise 36). We can think of the answer first through the sum of all digits in the integers from 0 to 999,999. We write 0 as 000000, 1 as 000001, 2 as 000002 and so on. Thus, we have a total of 10^6 numbers with the expected value of each digit being 4.5, since every digit occurs 10^5 times at each fixed position. Therefore, the sum of all digits would be $10^6 \cdot 4.5 \cdot 6 + 1 = 27,000,001$.

The zero string contributes nothing, and the additional 1 in the final calculation is the digit sum of 1,000,000.

Problem 10.37 (Exercise 37). Let $n \geq 1$ be an integer. How many positive integers less than 10^n are there, such that no two adjacent digits in their decimal representations are equal to each other?

Solution 10.37 (Exercise 37). For a number of length d , the first digit can be any integer from 1-9, so there are 9 choices. Every later digit can be any of the ten digits except the digit immediately before it, again giving 9 choices. Thus, there are 9^d numbers of length d with no equal adjacent digits.

Now we sum over all possible lengths from 1 to n . This gives the geometric series

$$\sum_{d=1}^n 9^d = \frac{9(9^n - 1)}{8}.$$

Therefore, this is the number of positive integers less than 10^n such that no two adjacent digits are equal.

Problem 10.38 (Exercise 41). According to the rules, a set of a volleyball game is won by the team that first scores 25 points with a minimum two-point advantage. After each point the result is written down, for example 0:1, 0:2, 1:2, 2:2, ... The first integer always shows points scored by the host team. Suppose that the final result of a set is 25 : n . How many ways can this result be reached

- (a) If $n \in \{0, 1, 2, \dots, 23\}$ is a fixed integer
- (b) If we only know that $n \leq 23$

Solution 10.38 (Exercise 41). For part (a), we know that the score had to be $24 : n$ before the host team scored the last point. There were $24 + n$ scoring events before that point, and we choose which n of them were scored by the away team. This gives $\binom{24+n}{n} = \binom{24+n}{24}$ ways. Every such sequence is allowed: before the last point the host has at most 24 points and the away team at most 23, so neither team could already have won the set. The last point is forced to be the host's.

For part (b), we sum the corrected expression over $0 \leq n \leq 23$. By the Hockey-Stick Identity,

$$\sum_{n=0}^{23} \binom{24+n}{24} = \binom{48}{25}.$$

Problem 10.39 (Exercise 42). Given $n \geq 0$ indistinguishable balls and $m \geq 1$ boxes numbered 1, 2, ..., m . How many ways can the balls be put into the boxes such that:

- (a) every box contains at least one ball?
- (b) there are exactly k balls in the first box, and every box contains at least one ball?
- (c) there are exactly k balls in the first box?
- (d) for any $k \in \{1, 2, 3, \dots, m\}$, the k -th box contains at least k balls.

Solution 10.39 (Exercise 42). All parameters are integers, and $k \geq 0$ when it is used. First take $m \geq 2$. (a) We know that m out of the n balls go into the boxes, leaving behind $n - m$ balls we can choose. Using the stars and bars method, we have $n - m$ balls and m boxes left which means we have $\binom{n-m+m-1}{m-1} = \binom{n-1}{m-1}$ ways in part a, provided $n \geq m$. If $n < m$, the answer is zero.

(b) We once again distribute m balls into the boxes and an additional $k - 1$ balls into the first box. This means we have used up $m + k - 1$ balls and have $n - m - k + 1$ balls left. Using stars and bars (only $m-1$ boxes this time), we get that there are $\binom{n-k-1}{m-2}$ ways of choosing the

balls, provided $k \geq 1$ and $n - k \geq m - 1$. Otherwise the answer is zero. The first box is fixed and receives none of the remaining balls.

(c) We give the first box k balls. Now, we have $n - k$ balls which we can distribute among the remaining $m - 1$ boxes. This is a total of $\binom{n+m-k-2}{m-2}$ ways to choose the balls using stars and bars, provided $0 \leq k \leq n$. Otherwise the answer is zero.

(d) We use up a total of $\frac{m(m+1)}{2}$ balls to fill in the requirement, since $1 + 2 + \dots + m = m(m+1)/2$. Thus, we are left with $n - \frac{m(m+1)}{2}$ balls among the remaining m boxes. Using stars and bars we get a total of $\binom{n - \frac{m(m+1)}{2} + m - 1}{m-1}$ ways to choose the balls, provided $n \geq m(m+1)/2$. Otherwise the answer is zero.

For $m = 1$, parts (a) and (d) have one distribution if $n \geq 1$, and zero otherwise. Part (b) has one distribution if $n = k \geq 1$, and zero otherwise. Part (c) has one distribution if $n = k \geq 0$, and zero otherwise.

Problem 10.40 (Exercise 43). Let $m \geq 1$ and $n_1, n_2, n_3 \geq 0$ be integers. How many ways are there to put n_1 blue balls, n_2 yellow balls and n_3 red balls into m boxes labeled $1, 2, 3, \dots, m$. Balls of same color are indistinguishable.

Solution 10.40 (Exercise 43). Take $m \geq 1$ and $n_1, n_2, n_3 \geq 0$ to be integers. We notice that the number of total ways to put all the balls in the boxes is the product of the number of ways to distribute each color separately among the boxes. Thus, if there are x balls of a color, there is a total of $\binom{x+m-1}{m-1}$ ways to arrange the balls of the colors. Plugging in n_1, n_2, n_3 , we get the total ways to be $\binom{n_1+m-1}{m-1} \binom{n_2+m-1}{m-1} \binom{n_3+m-1}{m-1}$ ways to put balls in boxes.

Problem 10.41 (Exercise 45). How many ways are there to put 8 red balls, 9 blue balls and 10 yellow balls into three boxes numbered 1, 2 and 3 such that any box contains at least one ball of each color.

Balls of the same color are indistinguishable.

Solution 10.41 (Exercise 45). Since 1 ball of each color first goes into each box, we are left with 5 red balls, 6 blue balls and 7 yellow balls among 3 boxes. This is a total of $\binom{7}{2} \binom{8}{2} \binom{9}{2} = 21168$ ways.

Problem 10.42 (Exercise 47). How many triples (x_1, x_2, x_3) of positive integers are there such that:

(a) $x_1 x_2 x_3 = 10^3$

(b) $x_1 x_2 x_3 = 10^3$ and $x_1 \leq x_2 \leq x_3$

Solution 10.42 (Exercise 47). (a). We notice that $x_1 x_2 x_3 = 2^3 5^3$. Now, this means that if $x_1 = 2^{a_1} * 5^{b_1}$, $x_2 = 2^{a_2} * 5^{b_2}$, $x_3 = 2^{a_3} * 5^{b_3}$, that $a_1 + a_2 + a_3 = 3$ and $b_1 + b_2 + b_3 = 3$, where all exponents are nonnegative integers. Thus, using stars and bars the total number of arrangements for powers of 2 is $\binom{5}{2} = 10$ and this is the same for powers of 5. Thus, the total ways is $10^2 = 100$.

(b) We can divide the cases from (a) into three parts: either $x_1 = x_2 = x_3$, $x_1 \neq x_2 \neq x_3 \neq x_1$ or neither. Now, the first case only happens when $x_1 = x_2 = x_3 = 10$. This is a total of 1 way. The third case happens when exactly two of x_1, x_2, x_3 equal each other. Let's assume that $x_1 = x_2$ WLOG. This means that $a_1 + a_2 \leq 3, b_1 + b_2 \leq 3$. The repeated value d must have $d^2 \mid 1000$, so $d \in \{1, 2, 5, 10\}$. Excluding the all-equal case $d = 10$ gives $2 \cdot 2 - 1 = 3$ cases, with sorted triples $(1, 1, 1000)$, $(2, 2, 250)$ and $(5, 5, 40)$. Multiplying it by 3 yields a total of 9 cases

here but we divide by 3 again since $x_1 \leq x_2 \leq x_3$. Thus, we have a total of 3 combinations of x_1, x_2, x_3 in this case. In the last case, we realize that there is a total of $100 - 1 - 9 = 90$ permutations left. Dividing by 6 yields a total of 15 combinations. Hence, we have a total of $1 + 3 + 15 = 19$ total combinations of x_1, x_2, x_3 .

Problem 10.43 (Exercise 48). Suppose that n distinct points are given on each side of a square. None of these points coincides with a vertex of the square. How many triangles are determined by these $4n$ points?

Solution 10.43 (Exercise 48). There are $\binom{4n}{3}$ ways to make a triangle with all the points on the square. However, this count includes the case where all three of the chosen points are on the same side of the square. Thus, we must subtract the number of ways to choose three points on the same side of the square. This can be calculated by $\binom{n}{3}$ for each side. Therefore, there are $\binom{4n}{3} - 4\binom{n}{3}$ triangles determined by these $4n$ points.

There are no other collinear triples: a line not lying along a side meets the boundary of a convex square in at most two points. Thus, the subtraction removes exactly all degenerate triples.

Problem 10.44 (Exercise 63 part (a)). Suppose that $m + n$ persons are in a line in front of a box office. Each of them intends to buy a ticket, and the ticket price is 5. Suppose that m of the persons in the line have 5 notes, and all of the remaining n persons have a 10 note. Assume also that $m \geq n$. How many ways can the line be rearranged, such that no one in the line with a 10 note will wait for change, if: there is no money in the cash register at the beginning? The people are distinct. Take m, n to be nonnegative integers with $m \geq n$.

Solution 10.44 (Exercise 63 part (a)). To consider the total number of arrangements, we first record only whether each person carries a 5 note or a 10 note. Call these letters U and D, respectively. There are $\binom{m+n}{n}$ unrestricted note-type patterns. For the cashier always to give change, every initial part of the line must have at least as many Us as Ds.

Now, we exclude the bad patterns. Read a pattern from the start, increasing a height by 1 at a U and decreasing it by 1 at a D. A bad pattern first reaches height -1 at some position. Interchange U and D in the initial part ending at that first visit. That part had one more D than U, so the new full pattern has $m + 1$ Us and $n - 1$ Ds.

Conversely, any pattern with $m + 1$ Us and $n - 1$ Ds ends at height $m - n + 2 \geq 2$, so it has a first visit to height 1. Interchanging U and D up to that visit gives a bad original pattern, whose first visit to height -1 is at the same position. These operations undo each other. Thus, when $n \geq 1$, the number of bad patterns is $\binom{m+n}{n-1}$.

Therefore, the number of good note-type patterns is

$$\binom{m+n}{n} - \binom{m+n}{n-1} = \frac{m-n+1}{m+1} \binom{m+n}{n}.$$

When $n = 0$, the only type pattern is good, and the second binomial coefficient is zero by our convention.

Finally, the people are distinct. We can arrange the m people with 5 notes in their positions in $m!$ ways, and the n people with 10 notes in $n!$ ways. Thus, the requested number of lines is

$$m!n! \left[\binom{m+n}{n} - \binom{m+n}{n-1} \right] = \frac{m-n+1}{m+1} (m+n)!.$$

Problem 10.45 (Exercise 67). Some chess masters and chess grandmasters took place in a chess tournament. All participants of the tournament played a game against each other. The number of games played by masters against the grandmasters was half the total number of games played

in the tournament. Prove that the total number of people in the tournament was a perfect square.

Solution 10.45 (Exercise 67). We have x masters and y grandmasters. Since half the games were between masters and grandmasters, the other half were played between only masters or only grandmasters. This means that $\binom{x}{2} + \binom{y}{2} = xy$. We can simplify this into

$$\begin{aligned}x^2 - x + y^2 - y &= 2xy, \\x^2 + y^2 - 2xy &= x + y, \\(x - y)^2 &= x + y.\end{aligned}$$

This means that $x + y$ is a perfect square and thus the number of people in the tournament is a perfect square.

Problem 10.46 (Exercise 69 part (a)). Let $n \geq 1$ be an integer and let ABCD be a square whose Cartesian coordinates are A(0,0), B(n,0), C(n,n) and D(0,n).

How many broken lines L are there if L starts at A, ends at C, and consists of $2n$ unit horizontal or vertical lattice steps? The lattice vertices have nonnegative integer coordinates. This is the rectilinear lattice-path interpretation of the question.

Solution 10.46 (Exercise 69 part(a)). We notice that the horizontal displacement is n and the vertical displacement is n . Let r, l, u, d be the numbers of right, left, up and down steps. Then $r - l = n$ and $u - d = n$, while $r + l + u + d = 2n$. Adding the displacement equations and comparing gives $l + d = 0$. Thus, there are exactly n right steps and n up steps, with no left or down steps.

We choose which n of the $2n$ step positions are right steps, and the remaining positions are up steps. Therefore, the total number of broken lines is $\binom{2n}{n}$.

Problem 10.47 (Exercise 69 part(b)). The square ABCD is partitioned into n^2 unit squares. How many ways can n indistinguishable chips be arranged in these unit squares, such that every row and every column contains exactly one chip?

Solution 10.47 (Exercise 69 part (b)). We put one chip in each column starting from Column 1 till Column n . We have n options for which row to put Chip 1. We then have $n-1$ options for which row to put Chip 2 in. This goes on until we have 1 option on which row to put Chip n in. Thus, we have $n!$ ways of arranging these chips and fulfilling these conditions.

11. ACKNOWLEDGEMENT

We are grateful to our mentor, Yaashaa Golovanov, and his programme, AMMOC, for offering us such a stimulating environment for the cultivation of necessary cognitive endurance and intellectual/mathematical development.