

CRUX MATHEMATICORUM

PRASANNA PAWAR, AMMOC - BENGALURU, INDIA

1. MATHEMATTIC

Problem 1.1 (MA276). Table-tennis balls come in boxes of 8 and in boxes of 15. For example, an order for 38 balls may be filled without breaking open any box by sending 2 boxes of 15 and 1 box of 8 while an order for 37 balls cannot be filled without breaking open a box. What is the maximum number of balls that cannot be bought without breaking open a box?

Solution 1.1. There are either 8 table-tennis balls in a box or there are 15 table-tennis balls in a box.

8 and 15 are co-prime numbers.

Therefore, to solve this problem we can use the **Chicken McNugget theorem** or the **Frobenius Coin theorem** which states that for any two co-prime numbers **m** and **n**, the greatest integer that cannot be written in the form $am + bn$ for non-negative integers a and b is :

$$mn - m - n$$

In our case , $m = 8$ and $n = 15$

This means that **a** is the number of boxes with 8 table-tennis balls and **b** is the number of boxes with 15 table-tennis balls

Therefore, the greatest number that cannot be written in the form $8a + 15b$ is

$$8 \times 15 - 8 - 15 = 97$$

In other words, the maximum number of balls that cannot be bought in boxes of 8 table-tennis balls and 15 table-tennis balls without breaking open a box is **97**

Problem 1.2 (MA277). Two congruent rectangles have a common vertex and overlap, as shown in the diagram. Find the area of the drawing that is not common to both rectangles.

Solution 1.2. Join the two vertices of the quadrilateral formed by the common area and call the line h .

The breadth of the rectangle is 8 units.

Let the length be $(x+2)$ units

In the common area, we get two right triangles with the hypotenuse h .

By applying the pythagoras theorem, we get:

$$8^2 + x^2 = h^2$$

$$4^2 + (x+2)^2 = h^2$$

$$\therefore 8^2 + x^2 = 4^2 + (x+2)^2$$

$$64 + x^2 = 16 + x^2 + 4 + 4x$$

$$44 = 4x$$

$$\therefore x=11$$

$$x+2=13$$

$\therefore$ The length of the rectangles is 13 units.

The area of one rectangle is :

$$\text{Length} \times \text{Breadth} = 13 \times 8 = 104$$

The common area is the sum of areas of the two right triangles formed :

$$\frac{1}{2} \times 8 \times x + \frac{1}{2} \times 4 \times (x+2)$$

$$= 4 \times 11 + 2 \times 13 = 44 + 26 = \mathbf{70}$$

As there are two rectangles, the area in the question which is not common to both rectangles is :

$$2 \times (\text{Area of a rectangle} - \text{Common area}) =$$

$$2 \times (104 - 70) = 2 \times 34 = \mathbf{68}$$

$\therefore$ The area of the drawing which is not common to both rectangles is 68

Problem 1.3 (MA278). The points A, B, C, D, E, F are collinear and the points P, Q, C, R, S are collinear, as shown. How many different triangles may be formed by using three of these given points as vertices?

Solution 1.3. The points A, B, C, D, E, F are collinear and the points P, Q, C, R, S are collinear.

Case 1: Take A as one of the vertices.

- If we say points A and P are vertices, there are **8** other points which can form triangles with points A and P.
 - If we say points A and Q are vertices, there are **7** other points which can form triangles with points A and Q such that they have not been counted earlier (for example, $\triangle APQ$ which has been counted in our first case of having A and P as the vertices) .
 - If we say points A and R are vertices, there are **6** other points which can form triangles with points A and R such that they have not been counted earlier.
 - If we say points A and S are vertices, there are **5** other points which can form triangles with points A and S such that they have not been counted earlier.
- $\therefore$ When we already take A as one of the vertices there are $8 + 7 + 6 + 5 = \mathbf{26}$ unique triangles.

Case 2: Take B as one of the vertices.

- If we say points B and P are vertices, there are **7** other points which can form triangles with points B and P such that they have not been counted earlier.
 - If we say points B and Q are vertices, there are **6** other points which can form triangles with points B and Q such that they have not been counted earlier.
 - If we say points B and R are vertices, there are **5** other points which can form triangles with points B and R such that they have not been counted earlier.
 - If we say points B and S are vertices, there are **4** other points which can form triangles with points B and S such that they have not been counted earlier.
- $\therefore$ When we already take B as one of the vertices there are $7 + 6 + 5 + 4 = \mathbf{22}$ unique triangles.

Case 3: Take D as one of the vertices.

- If we say points D and P are vertices, there are **6** other points which can form triangles with points D and P such that they have not been counted earlier.
 - If we say points D and Q are vertices, there are **5** other points which can form triangles with points D and Q such that they have not been counted earlier.
 - If we say points D and R are vertices, there are **4** other points which can form triangles with points D and R such that they have not been counted earlier.
 - If we say points D and S are vertices, there are **3** other points which can form triangles with points D and S such that they have not been counted earlier.
- $\therefore$ When we already take D as one of the vertices there are $6 + 5 + 4 + 3 = \mathbf{18}$ unique triangles.

Case 4: Take E as one of the vertices.

- If we say points E and P are vertices, there are **5** other points which can form triangles with points E and P such that they have not been counted earlier.
 - If we say points E and Q are vertices, there are **4** other points which can form triangles with points E and Q such that they have not been counted earlier.
 - If we say points E and R are vertices, there are **3** other points which can form triangles with points E and R such that they have not been counted earlier.
 - If we say points E and S are vertices, there are **2** other points which can form triangles with points E and S such that they have not been counted earlier.
- ∴ When we already take E as one of the vertices there are $5 + 4 + 3 + 2 = 14$ unique triangles.

Case 5: Take F as one of the vertices.

- If we say points F and P are vertices, there are **4** other points which can form triangles with points F and P such that they have not been counted earlier.
 - If we say points F and Q are vertices, there are **3** other points which can form triangles with points F and Q such that they have not been counted earlier.
 - If we say points F and R are vertices, there are **2** other points which can form triangles with points F and R such that they have not been counted earlier.
 - If we say points F and S are vertices, there are **1** other points which can form triangles with points F and S such that they have not been counted earlier.
- ∴ When we already take E as one of the vertices there are $4 + 3 + 2 + 1 = 10$ unique triangles.

By adding all of them, we get :

$$26 + 22 + 18 + 14 + 10 = 90$$

∴ The number of different triangles which can be formed by using any three of these given points as vertices is **90**

Problem 1.4 (MA279). A 4-digit number is a perfect square. When 1 is added to each of the digits, another 4-digit perfect square is formed. What number is the smaller of these perfect squares?

Solution 1.4. The smallest 4 digit perfect square number is -

$$1024 = 32^2$$

The largest 4 digit perfect square number is -

$$9801 = 99^2$$

Therefore, the four digit number is a square of a two digit number.

According to the problem, when $\overline{AB}^2 = \overline{CDEF}$

then $(C+1)(D+1)(E+1)(F+1)$ is also a perfect square

Let $\overline{AB} = 45$

$$45^2 = 2025$$

Therefore $\overline{CDEF} = 2025$

$$(C+1)(D+1)(E+1)(F+1) = 3136$$

3136 is a perfect square number as $56^2 = 3136$

The smaller of these perfect squares is **2025**

Problem 1.5 (MA 280). Prove that every odd integer n lies exactly halfway between a multiple of 3 and a multiple of 4 where $n \geq 5$.

Solution 1.5. Consider the number n

If n lies exactly halfway between a multiple of 3 and a multiple of 4

$$n = \frac{3m+4p}{2} \text{ for some integers } m \text{ and } p.$$

$$2n = 3m + 4p$$

As $2n$ and $4p$ are even, $3m$ has to be even as the difference ($3m = 2n - 4p$) of two even numbers is an even number.

For $3m$ to be even, m has to be even.

Therefore, let $m = 2x$

$$\text{So } 2n = 3 \times (2x) + 4p$$

$$n = \frac{3 \times 2x + 4p}{2}$$

$$n = 3x + 2p$$

Now x cannot be even because if x is even :

$$x = 2y$$

$$3m = 2 \times (2y) = 4y$$

But we have to prove that n lies halfway between a multiple of 3 and a multiple of 4 , therefore we do not consider another multiple of 4 .

Therefore, x is always odd .

Therefore n is always odd as the sum of an odd number and an even number is always odd .

As $x \neq 0$ and $p \neq 0$ the minimum value of n is $(3 \times 1 + 2 \times 1)$, which is 5.

Therefore all odd integers n greater than or equal to 5 lie between a multiple of 3 and a multiple of 4.

Hence Proved.

2. OLYMPIAD CORNER

Problem 2.1 (OC689). The product of five consecutive positive integers is 120 times greater than $\overline{ABABAB}$, where A and B are non-zero digits. What is the largest of those five numbers?

Solution 2.1. Let the five consecutive integers be n , $(n+1)$, $(n+2)$, $(n+3)$ and $(n+4)$

We know that :

$$n \times (n+1) \times (n+2) \times (n+3) \times (n+4) = 120 \times \overline{ABABAB}$$

$$\frac{n \times (n+1) \times (n+2) \times (n+3) \times (n+4)}{120} = \overline{ABABAB}$$

We can get an integer in the form $\overline{ABABAB}$ only when $n = 35$

$$\frac{35 \times 36 \times 37 \times 38 \times 39}{120} = \frac{69090840}{120} = 575757$$

575757 is in the form $\overline{ABABAB}$ which means $n = 35$ satisfies this condition

The five numbers are 35, 36, 37, 38 and 39

Therefore the largest of those 5 numbers is 39

Problem 2.2 (OC690). Let ABC be an equilateral triangle whose area is 36cm^2 , and EFD be an isosceles triangle with $EF = FD$. The point F is the centre of triangle ABC, and the points B and C are the midpoints of EF and FD, respectively, as shown in the diagram. If $BG \perp EF$ and $CH \perp DF$, what is the area, in cm^2 , of the shaded region?

Solution 2.2. Let the area of a triangle XYZ be denoted as $\text{area}[\triangle XYZ]$ in this solution. As $\triangle ABC$ is an equilateral triangle $AB = BC = AC$

As F is the centre of an equilateral triangle, $AF = BF = CF$

$\therefore$ By SSS criterion :

$$\triangle AFB \cong \triangle BFC \cong \triangle CFA$$

Which means all of them have equal areas as well.

$$\text{area}[\triangle AFB] + \text{area}[\triangle BFC] + \text{area}[\triangle CFA] = \text{area}[\triangle ABC]$$

$$\therefore 3 \times \text{area}[\triangle AFB] = 3 \times \text{area}[\triangle BFC] = 3 \times \text{area}[\triangle CFA] = 36\text{cm}^2$$

$$\text{area}[\triangle AFB] = \text{area}[\triangle BFC] = \text{area}[\triangle CFA] = 12\text{cm}^2$$

Now, in $\triangle EFD$, B is the midpoint of EF and C is the midpoint of FD.

Using the midpoint theorem, which states that : In a triangle, a line joining the midpoints of two sides is parallel to the third line and has the half of the length of the third line.

$$\therefore BC \parallel ED$$

In a triangle, when we draw a line parallel to the third line, the triangle formed is similar to the larger triangle.

In our case this means : $\triangle BFC \sim \triangle EFD$

In similar triangles, the ratio of the areas of the two triangles is the square of the ratio of two corresponding sides.

$$\therefore \left(\frac{BF}{EF}\right)^2 = \frac{\text{area}[\triangle BFC]}{\text{area}[\triangle EFD]}$$

As B is the midpoint of EF, $\frac{BF}{EF} = \frac{1}{2}$

$$\therefore \left(\frac{1}{2}\right)^2 = \frac{\text{area}[\triangle BFC]}{\text{area}[\triangle EFD]}$$

$$\frac{1}{4} = \frac{12}{\text{area}[\triangle EFD]}$$

$$\therefore \text{area}[\triangle EFD] = 48\text{cm}^2$$

Construction : $FI \perp ED$

Now, $EB = BF$ (because B is the midpoint of EF) .

$$\angle EBG = \angle DCH = 90^\circ$$

BG is common

$\therefore$ By the SAS criterion, we can say that : $\triangle EBG \cong \triangle FBG$

Similarly, $\triangle DCH \cong \triangle FCH$

As $\triangle FED$ is isosceles, $\angle FED = \angle FDE$

$$\frac{FE}{2} = \frac{FD}{2} \text{ or in other words, } BE = CD.$$

$$\angle EBG = \angle DCH = 90^\circ$$

$\therefore$ by ASA criterion, $\triangle EBG \cong \triangle DCH$.

$$\therefore \triangle EBG \cong \triangle FBG \cong \triangle FCH \cong \triangle DCH$$

This means, the areas of these four triangles are equal as well.

$$\angle FIG + \angle FBG = 90^\circ + 90^\circ = 180^\circ$$

$\therefore$ BFIG is a cyclic quadrilateral as the sum of opposite angles is 180° .

$$\angle FIIH + \angle FCH = 90^\circ + 90^\circ = 180^\circ$$

$\therefore$ CFIH is a cyclic quadrilateral as the sum of opposite angles is 180° .

In a cyclic quadrilateral, after connecting two vertices, if we get two right triangles, the areas of these two right triangles are equal.

This is a direct consequence of the fact that in a circle, if the diameter subtends two right angles on the circumference, the areas of the two right triangles formed are equal.

$$\therefore \text{area}[\triangle FIG] = \text{area}[\triangle FBG] \text{ and } \text{area}[\triangle FIIH] = \text{area}[\triangle FCH]$$

By using the facts we have concluded above, we can say that:

$$\text{area}[\triangle EBG] = \text{area}[\triangle FBG] = \text{area}[\triangle FIG] = \text{area}[\triangle FIIH] = \text{area}[\triangle FCH] = \text{area}[\triangle DCH] = x \text{ cm}^2$$

$$\text{also } \text{area}[\triangle EBG] + \text{area}[\triangle FBG] + \text{area}[\triangle FIG] + \text{area}[\triangle FIIH] + \text{area}[\triangle FCH] + \text{area}[\triangle DCH] = \text{area}[\triangle FED]$$

$$\therefore 6x = 48$$

$$x = 8$$

The area of the shaded region is is :

$$\text{area}[\triangle AFB] + \text{area}[\triangle CFA] + \text{area}[\triangle EBG] + \text{area}[\triangle CHD]$$

$$= 12 + 12 + 8 + 8$$

$$= 40$$

$\therefore$ The area of the shaded region, in cm^2 , is 40 cm^2