

MATHEMATICAL REFLECTIONS

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1. JUNIOR SECTION

Problem 1.1 (J661 2024 - Issue 3). Find all positive integers n for which $\frac{(n^2+1)!+(n^2)!+(n^2-1)!}{(n+1)!} = (n+3)^2$.

Solution 1.1 (J661).

$$\frac{(n^2+1)!+(n^2)!+(n^2-1)!}{(n+1)!} = (n+3)^2$$

$$(n^2+1)!+(n^2)!+(n^2-1)! = ((n+1)!(n+3)^2$$

Now we take $(n^2-1)!$ as common

$$\begin{aligned}(n^2-1)!((n^2+1)(n^2)+n^2+1) &= ((n+1)!(n+3)^2 \\ (n^2-1)!(n^4+n^2+n^2+1) &= ((n+1)!(n+3)^2 \\ (n^2-1)!(n^4+2n^2+1) &= ((n+1)!(n+3)^2 \\ (n^2-1)!((n^2+1)^2) &= ((n+1)!(n+3)^2 \\ (n^2-1)!((n^2+1)^2) &= (((n+2)-1)!((n+2)+1)^2\end{aligned}$$

Let

$$n^2 = a$$

and

$$n+2 = b$$

We can rewrite the equation as

$$(a-1)!((a+1)^2) = ((b-1)!((b+1)^2$$

Case 1 :

$$a > b$$

$$(a-1)! > (b-1)!$$

$$(a+1)^2 > (b+1)^2$$

Therefore,

$$(a-1)!((a+1)^2) > (b-1)!((b+1)^2)$$

This is false as we know that,

$$(a-1)!((a+1)^2) = ((b-1)!((b+1)^2$$

$\therefore a$ cannot be greater than b

Case 2 : Assume that

$$a < b$$

$$(a - 1)! < (b - 1)!$$

$$(a + 1)^2 < (b + 1)^2$$

Therefore,

$$(a - 1)!((a + 1)^2) < (b - 1)!((b + 1)^2)$$

This is false as we know that,

$$(a - 1)!((a + 1)^2) = ((b - 1)!)((b + 1)^2)$$

$\therefore$ a cannot be lesser than b

The only possibility is : $a = b$

As $a = b$ we get

$$n^2 = n + 2$$

$$n^2 - n - 2 = 0$$

$$(n + 1)(n - 2) = 0$$

The two values of n are , $n = -1$ and $n = 2$

As we have been asked to find out a positive value of n,

$n = -1$ cannot be a solution

$\therefore n = 2$

Problem 1.2 (J667 2024 - Issue 4). Prove that the sum of squares of two twin primes can be written as the sum of squares of three positive integers.

Lemma 1.1. To solve this problem we will first prove that all twin primes except (3,5) can be written in the form $6k+1$, $6k-1$ where k is a positive integer.

Proof: An integer can be written as $6k$, $6k+1$, $6k+2$, $6k+3$, $6k+4$ or $6k+5$.

$6k$ is never prime because $6|6k$

$6k+2$ is never prime because $2|2(3k + 1)$

$6k+3$ is never prime because $3|3(2k + 1)$

$6k+4$ is never prime because $2|2(3k + 2)$

$\therefore$ only $6k+1$ and $6k+5$ can be primes.

$6k+5$ is basically $6(k+1)-1$.

If it is true for $6(k+1)-1$, it should be true for $6k-1$ as well as it is in the same form for an integer k .

$\therefore$ All twin primes except $(3,5)$ can be written in the form $6k+1$, $6k-1$ because $6k+1$ and $6k-1$ could be primes for certain integers k and the difference between them is 2 which is the definition of twin primes

Solution 1.2. Using Lemma 1.1 we proceed with the problem.

First we prove it for twin primes $(3,5)$:

$$3^2 + 5^2 = 9 + 25 = 34$$

$$34 = 16 + 9 + 9 = 4^2 + 3^2 + 3^2 \text{ which is in the form of sum of squares of three positive integers.}$$

$\therefore$ Sum of squares of 3 and 5 can be written as sum of three positive integers.

Now we proceed with this problem using the fact that all twin primes except $(3,5)$ can be written as $6k-1$ and $6k+1$.

$\therefore$ The sum of their squares can be written as :

$$\begin{aligned}(6k-1)^2 + (6k+1)^2 &= (36k^2 + 1 - 12k) + (36k^2 + 1 + 12k) \\ &= 72k^2 + 2 \\ &= 2 \times (36k^2 + 1)\end{aligned}$$

As $(36k^2 + 1)$ is odd, it will not be divisible by 2.

$\therefore 2 \times (36k^2 + 1)$ is not divisible by 4.

The **Legendre's Three Square Theorem** states that a natural number n can be represented as a sum of three squares of integers if and only if n is not of the form :

$$4^a(8b+7)$$

Using this theorem we can conclude that :

All integers in the form $(6k-1)^2 + (6k+1)^2$, where k is an integer, can be written as sum of squares of three integers because $4^a(8b+7)$ is divisible by 4 and $2 \times (36k^2 + 1)$ is not divisible by 4.

This means $2 \times (36k^2 + 1)$ cannot be written in the form $4^a(8b+7)$

$\therefore$ Sum of squares of two twin primes can be written as the sum of squares of three positive integers.

HENCE PROVED.

Problem 1.3 (J669 2024 - Issue 4). Let a, b, c be positive real numbers. Prove that

$$\frac{a+b}{a^2-ab+b^2} + \frac{b+c}{b^2-bc+c^2} + \frac{c+a}{c^2-ca+a^2} + \frac{ab+bc+ca}{abc} \geq \frac{3(a+b+c)^2}{a^3+b^3+c^3}.$$

Solution 1.3. Using the Titu's Lemma which states that:

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \dots + \frac{a_n^2}{b_n} \geq \frac{(a_1+a_2+\dots+a_n)^2}{b_1+b_2+\dots+b_n}.$$

We can say that:

$$\begin{aligned}\frac{a^2}{a^3} + \frac{b^2}{b^3} + \frac{c^2}{c^3} &\geq \frac{(a+b+c)^2}{a^3+b^3+c^3} \\ \frac{1}{a} + \frac{1}{b} + \frac{1}{c} &\geq \frac{(a+b+c)^2}{a^3+b^3+c^3} \\ \frac{ab+ac+bc}{abc} &\geq \frac{(a+b+c)^2}{a^3+b^3+c^3} \\ \text{Multiply Both sides by 3:} \\ \frac{3(ab+ac+bc)}{abc} &\geq \frac{3(a+b+c)^2}{a^3+b^3+c^3}\end{aligned}$$

With equality appearing at $a=b=c=1$

Using the AM-GM inequality

$$\begin{aligned}\frac{a^2+b^2}{2} &\geq \sqrt{a^2b^2} \\ a^2+b^2 &\geq 2ab \\ a^2-ab+b^2 &\geq ab\end{aligned}$$

$$\frac{1}{ab} \geq \frac{1}{a^2-ab+b^2}$$

Multiply $a+b$ on both sides

$$\begin{aligned}\frac{a+b}{ab} &\geq \frac{a+b}{a^2-ab+b^2} \\ \therefore \frac{1}{a} + \frac{1}{b} &\geq \frac{a+b}{a^2-ab+b^2}\end{aligned}$$

Similarly

$$\begin{aligned}\therefore \frac{1}{b} + \frac{1}{c} &\geq \frac{b+c}{b^2-bc+c^2} \\ \text{and } \therefore \frac{1}{c} + \frac{1}{a} &\geq \frac{c+a}{c^2-ca+a^2}\end{aligned}$$

By adding these three inequalities we get.

$$\begin{aligned}\frac{2}{a} + \frac{2}{b} + \frac{2}{c} &\geq \frac{a+b}{a^2-ab+b^2} + \frac{b+c}{b^2-bc+c^2} + \frac{c+a}{c^2-ca+a^2} \\ \frac{2ab+2bc+2ac}{abc} &\geq \frac{a+b}{a^2-ab+b^2} + \frac{b+c}{b^2-bc+c^2} + \frac{c+a}{c^2-ca+a^2}\end{aligned}$$

Now we add $\frac{ab+bc+ac}{abc}$ on both sides and get

$$\frac{3ab+3bc+3ac}{abc} \geq \frac{a+b}{a^2-ab+b^2} + \frac{b+c}{b^2-bc+c^2} + \frac{c+a}{c^2-ca+a^2} + \frac{ab+bc+ac}{abc}$$

With equality appearing at $a=b=c=3$.

As the equality with $\frac{3ab+3bc+3ac}{abc}$ appears at greater values for a, b and c in $\frac{a+b}{a^2-ab+b^2} + \frac{b+c}{b^2-bc+c^2} + \frac{c+a}{c^2-ca+a^2} + \frac{ab+bc+ac}{abc}$ than $\frac{3(a+b+c)^2}{a^3+b^3+c^3}$, we can say that

$$\frac{a+b}{a^2-ab+b^2} + \frac{b+c}{b^2-bc+c^2} + \frac{c+a}{c^2-ca+a^2} + \frac{ab+bc+ca}{abc} \geq \frac{3(a+b+c)^2}{a^3+b^3+c^3}$$

This inequality is verified when we put the value $a=1$, $b=2$ and $c=3$ in the inequality and get

the value of $\frac{a+b}{a^2-ab+b^2} + \frac{b+c}{b^2-bc+c^2} + \frac{c+a}{c^2-ca+a^2} + \frac{ab+bc+ca}{abc}$ as $\frac{173}{42}$ or approximately 4.12 and the value of $\frac{3(a+b+c)^2}{a^3+b^3+c^3}$ as 3.

Problem 1.4 (J672 2024 - Issue 4). Let ABCD be a convex quadrilateral such that

$$(AB - BC)^2 + (CD - DA)^2 = 2(AB \times AD - AC \times BD + BC \times CD).$$

Prove that the statements

- (a) ABCD is cyclic
 - (b) ABCD is circumscribable
- are equivalent.

Solution 1.4. Let us call the sides:

AB = a

BC = b

CD = c

DA = d

AC = e

BD = f

∴ The equation becomes :

$$(1) \quad (a - b)^2 + (c - d)^2 = 2ad - 2ef + 2bc$$

Now we prove that the statements are equivalent.

Case 1: ABCD is cyclic.

We use the Ptolemy's theorem which states that:

In a cyclic quadrilateral ABCD with side lengths a, b, c, d and diagonals e, f :

$ac + bd = ef$.

By replacing this value of ef in Equation(1) we get

$$(a - b)^2 + (c - d)^2 = 2ad - 2(ac + bd) + 2bc$$

$$(a - b)^2 + (c - d)^2 = 2ad - 2ac - 2bd + 2bc$$

This is only possible if $a+c = b+d$ because that would mean $a-b = c-d$ and the equation becomes

$$\begin{aligned} 2(c-d)^2 &= 2ad - 2ac - 2bd + 2bc \\ 2(c-d)^2 &= 2(ad - ac - bd + bc) \\ (c-d)^2 &= ad - ac - bd + bc \\ (c-d)^2 &= a(d-c) - b(d-c) \\ (c-d)^2 &= (d-c)(a-b) \end{aligned}$$

Now by using the fact that $(x-y)^2 = (y-x)^2 = x^2 + y^2 - 2xy$, we can say that $(c-d)^2 = (d-c)^2$

$$\begin{aligned} \therefore (d-c)^2 &= (d-c)(a-b) \\ d-c &= a-b \\ b+d &= c+a \end{aligned}$$

Which is true as we have already stated

Therefore, the equation $a+c = b+d$ is true. The Pitot Theorem states that in a circumscribable quadrilateral the sum of opposite sides is equal.

This basically means in a circumscribable quadrilateral $a+c = b+d$.

$\therefore$ When ABCD is a cyclic quadrilateral, its properties for circumscribable quadrilateral hold true as well.

Now we prove it the other way.

Case 2 : ABCD is circumscribable.

We use the Pitot's theorem which states that in a circumscribable quadrilateral, the sums of two opposite sides are equal. This means in our quadrilateral

$$\begin{aligned} a+c &= b+d \\ \text{which means } a-b &= d-c \end{aligned}$$

As we proved above : $(c-d)^2 = (d-c)^2$

$\therefore (a-b)^2 = (d-c)^2 = (c-d)^2$

Therefore, Equation(1) becomes

$$\begin{aligned} 2(c-d)^2 &= 2(ad - ef + bc) \\ (c-d)^2 &= ad - ef + bc \end{aligned}$$

This would be true only if $ef = ac + bd$, because then the equation becomes

$$\begin{aligned}(c-d)^2 &= ad - (ac + bd) + bc \\(c-d)^2 &= ad - ac - bd + bc \\(c-d)^2 &= -a(c-d) + b(c-d) \\(c-d)^2 &= (b-a)(c-d) \\c-d &= b-a \\c+a &= b+d\end{aligned}$$

Which is, as we stated above, what the Pitot's Theorem states.

Therefore, the claim that $ef = ac + bd$ is true.

The Ptolemy's Theorem states that in a cyclic quadrilateral : $ef = ac + bd$.

$\therefore$ When ABCD is a circumscribable quadrilateral, it's properties for cyclic quadrilateral hold true as well.

Therefore, the statements

- (a) ABCD is cyclic
 - (b) ABCD is circumscribable
- are equivalent.

HENCE PROVED

Problem 1.5 (J673 2024 - Issue 5). Let a be a positive real number. Prove that $(a-1)\sqrt{a} > 1$ if and only if $a^2(\sqrt{a}-1) > 1$.

Solution 1.5. By simplifying $(a-1)\sqrt{a} > 1$.

$$\begin{aligned}(a-1)\sqrt{a} &> 1 \\a\sqrt{a} - \sqrt{a} &> 1 \\a\sqrt{a} &> 1 + \sqrt{a}\end{aligned}$$

Now we multiply both sides by a

$$a^2\sqrt{a} > a + a\sqrt{a}$$

From our second inequality we get :

$$\begin{aligned}a^2(\sqrt{a}-1) &> 1 \\a^2\sqrt{a} &> 1 + a^2\end{aligned}$$

This is only possible if $a \geq 1.76$

We can substitute the same value of a in the first inequality and see that we get :

$$(1.76-1)(\sqrt{1.76}) = 1.008$$

which is greater than 1.

Therefore $a^2(\sqrt{a}-1) > 1$ being true implies that $(a-1)\sqrt{a} > 1$ is true. **Hence Proved.**

Problem 1.6 (J679 2024 - Issue 6). Let n be a positive integer. Find a representation of $N = 6^{8n} + 6^{4n+1} + 1$ as the sum of the squares of three distinct positive integers

Solution 1.6.

$$\begin{aligned} N &= 6^{8n} + 6^{4n+1} + 1 \\ N &= (6^{4n})^2 + 6 \times 6^{4n} + 1 \\ N &= (6^{4n} + 3)^2 - 9 + 1 \\ N &= (6^{4n} + 3)^2 - 8 \\ N &= (6^{4n} + 3)^2 - 2^2 - 2^2 \end{aligned}$$

Problem 1.7 (J684 2024 - Issue 6). Prove that there are no rational numbers x, y such that

$$2x^2 + 12y^2 = 1.$$

Solution 1.7.

$$2x^2 + 12y^2 = 1$$

As x and y are rational numbers we can say that there exist coprime a and b and coprime c and d such that

$$\begin{aligned} x &= \frac{a}{b}, y = \frac{c}{d} \\ 2\left(\frac{a}{b}\right)^2 + 12\left(\frac{c}{d}\right)^2 &= 1 \\ \frac{2a^2d^2}{b^2d^2} + \frac{12b^2c^2}{b^2d^2} &= 1 \\ 2a^2d^2 + 12b^2c^2 &= b^2d^2 \\ b^2d^2 - 2a^2d^2 - 12b^2c^2 &= 0 \end{aligned}$$

Consider this in modulus 4 now.

$$b^2d^2 - 2a^2d^2 - 12b^2c^2 \equiv 0 \pmod{4}$$

As 4 divides 12

$$\begin{aligned} b^2d^2 - 2a^2d^2 &\equiv 0 \pmod{4} \\ (b^2 - 2a^2)d^2 &\equiv 0 \pmod{4} \\ b^2 &\equiv 2a^2 \pmod{4} \end{aligned}$$

For this to be possible, both a and b would have to be even. This contradicts the fact that a and b are coprime. Therefore, by contradiction, we can conclude that there are no rational numbers x, y such that $2x^2 + 12y^2 = 1$.

Problem 1.8 (J691 2025 - Issue 2). Let a, b, c be real numbers such that

$$a^2(b+c) + b^2(c+a) + c^2(a+b) = 3(a+b+c-1)$$

and $a+b+c \neq 0$. Prove that $ab+bc+ca = 3$ if and only if $abc = 1$.

Solution 1.8. Let $d = a+b+c$, $e = ab+bc+ca$, and $f = abc$. First we rewrite the given equation in terms of d, e , and f .

Expand the LHS:

$$\begin{aligned} a^2(b+c) + b^2(c+a) + c^2(a+b) &= a^2(d-a) + b^2(d-b) + c^2(d-c). \\ d(a^2 + b^2 + c^2) - (a^3 + b^3 + c^3). \end{aligned}$$

Now express $a^2 + b^2 + c^2$ and $a^3 + b^3 + c^3$ using algebraic identities:

$$a^2 + b^2 + c^2 = d^2 - 2e,$$

$$a^3 + b^3 + c^3 = d^3 - 3de + 3f \quad (\text{derived from expanding } (a + b + c)^3).$$

The original equation $\text{LHS} = 3(d - 1)$ becomes:

$$de - 3f = 3d - 3.$$

Assume $e = 3$. Substituting into the equation:

$$d \cdot 3 - 3f = 3d - 3 \implies -3f = -3 \implies f = 1.$$

Therefore, $abc = 1$.

Assume $f = 1$. Substituting into the equation:

$$de - 3 = 3d - 3 \implies de = 3d.$$

Since $d \neq 0$, divide both sides by d to get $e = 3$. Hence, $ab + bc + ca = 3$.

This proves $ab + bc + ca = 3 \iff abc = 1$.

Problem 1.9 (J692 2025 - Issue 2). Given positive real numbers a, b, c such that $a + b + c + 2 = abc$ and $t = ab + bc + ca$, we need to prove that:

$$\sqrt{t + a^2} + \sqrt{t + b^2} + \sqrt{t + c^2} \leq t.$$

Solution 1.9.

$$\sqrt{t + a^2} = \sqrt{(a + b)(a + c)}, \quad \sqrt{t + b^2} = \sqrt{(b + a)(b + c)}, \quad \sqrt{t + c^2} = \sqrt{(c + a)(c + b)}.$$

$$t + a^2 = ab + bc + ca + a^2 = a(a + b + c) + bc = (a + b)(a + c).$$

For any $a, b, c > 0$, by AM-GM

$$\sqrt{(a + b)(a + c)} \leq \frac{(a + b) + (a + c)}{2} = \frac{2a + b + c}{2}.$$

Similarly,

$$\sqrt{(b + a)(b + c)} \leq \frac{2b + a + c}{2}, \quad \sqrt{(c + a)(c + b)} \leq \frac{2c + a + b}{2}.$$

Adding these inequalities:

$$\sqrt{(a + b)(a + c)} + \sqrt{(b + a)(b + c)} + \sqrt{(c + a)(c + b)} \leq \frac{4a + 4b + 4c}{2} = 2(a + b + c).$$

From the given condition $a + b + c + 2 = abc$, we have:

$$a + b + c = abc - 2.$$

Substitute this into $2(a + b + c)$:

$$2(a + b + c) = 2(abc - 2).$$

We need to show $2(a + b + c) \leq t$. Using $abc = a + b + c + 2$:

$$2(abc - 2) = 2(a + b + c) \leq ab + bc + ca = t.$$

From previous steps

$$\sqrt{t + a^2} + \sqrt{t + b^2} + \sqrt{t + c^2} \leq 2(a + b + c) \leq t.$$

Therefore, the inequality is proven:

$$\boxed{\sqrt{t+a^2} + \sqrt{t+b^2} + \sqrt{t+c^2} \leq t}.$$

Problem 1.10 (J693 2025 - Issue 2). Let a, b, c be positive real numbers such that $a + b + c = 3abc$, prove that

$$a^2 + b^2 + c^2 + 3 \geq 2(ab + bc + ca).$$

Solution 1.10. Let $t = abc$. The given condition becomes $a + b + c = 3t$. By the AM-GM inequality:

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc} \implies t \geq 1.$$

Using the identity $a^2 + b^2 + c^2 = (a+b+c)^2 - 2(ab+bc+ca)$:

$$a^2 + b^2 + c^2 + 3 = (3t)^2 - 2(ab+bc+ca) + 3 = 9t^2 - 2(ab+bc+ca) + 3.$$

Substitute into the original inequality:

$$9t^2 - 2(ab+bc+ca) + 3 \geq 2(ab+bc+ca).$$

$$9t^2 + 3 \geq 4(ab+bc+ca).$$

By AM-GM inequality for $ab+bc+ca$:

$$ab+bc+ca \geq 3\sqrt[3]{(abc)^2} = 3t^{2/3}.$$

Substitute this

$$9t^2 + 3 \geq 12t^{2/3}.$$

Let $s = t^{2/3}$ since $t \geq 1$, we have $s \geq 1$:

$$9s^3 + 3 \geq 12s \implies 3s^3 + 1 \geq 4s.$$

For $s \geq 1$, the polynomial $3s^3 - 4s + 1$ is non-negative because:

$$3(1)^3 - 4(1) + 1 = 0,$$

$$\frac{d}{ds}(3s^3 - 4s + 1) = 9s^2 - 4 \geq 5 > 0 \quad (\text{for } s \geq 1).$$

Thus, $3s^3 + 1 \geq 4s$ is true for all $s \geq 1$, and equality occurs when $a = b = c = 1$. Therefore:

$$\boxed{a^2 + b^2 + c^2 + 3 \geq 2(ab + bc + ca)}.$$

Problem 1.11 (J697 2025 - Issue 3). Jimmy plays with numbers and sees that a certain four-digit number $\overline{abcd}$ is equal to $(\overline{ab} + \overline{cd})^2$. Find all four-digit numbers with this property.

Solution 1.11. We seek all four-digit numbers $\overline{abcd}$ such that:

$$\overline{abcd} = (\overline{ab} + \overline{cd})^2$$

where $\overline{ab} = 10a + b$ and $\overline{cd} = 10c + d$.

Let $x = ab$ (a two-digit integer, $10 \leq x \leq 99$) and $y = cd$ (an integer, $0 \leq y \leq 99$). The equation becomes:

$$100x + y = (x + y)^2$$

$$(x + y)^2 - 100x - y = 0$$

Let $s = x + y$. Then:

$$s^2 - 100x - (s - x) = 0 \implies s^2 - s - 99x = 0$$

Solving for x :

$$x = \frac{s^2 - s}{99}$$

Since x must be an integer between 10 and 99, we check values of s such that $s^2 - s$ is divisible by 99. Trying s in the range $32 \leq s \leq 99$ (since $s \geq \sqrt{1000} \approx 31.62$ and $s \leq 99 + 99 = 198$), but smaller s values are more likely to give us a valid x :

- For $s = 45$:

$$x = \frac{45^2 - 45}{99} = \frac{2025 - 45}{99} = \frac{1980}{99} = 20$$

Then $y = s - x = 25$, giving the number $\overline{abcd} = 100 \times 20 + 25 = 2025$.

- For $s = 55$:

$$x = \frac{55^2 - 55}{99} = \frac{3025 - 55}{99} = \frac{2970}{99} = 30$$

Then $y = s - x = 25$, giving the number $\overline{abcd} = 100 \times 30 + 25 = 3025$.

- For $s = 99$:

$$x = \frac{99^2 - 99}{99} = \frac{9801 - 99}{99} = \frac{9702}{99} = 98$$

Then $y = s - x = 1$, giving the number $\overline{abcd} = 100 \times 98 + 01 = 9801$.

No other values of s yield integer x in the required range. Thus, the four-digit numbers satisfying the given condition are:

2025, 3025, 9801

Problem 1.12 (J704 2025 - Issue 4). Let a, b, c be positive real numbers such that $a + b + c = 6$. Prove that

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} + \frac{3}{2(a^2 + b^2 + c^2)} \geq \frac{7}{8}.$$

Solution 1.12. Assume without loss of generality that $b = c$. $a + b + c = 6$

$$a + 2b = 6$$

$$\text{so } a = 6 - 2b$$

$$\text{and } 0 < b < 3.$$

LHS:

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} = \frac{1}{6-b} + \frac{1}{2b} + \frac{1}{6-b} = \frac{2}{6-b} + \frac{1}{2b},$$

$$a^2 + b^2 + c^2 = (6 - 2b)^2 + b^2 + b^2 = 36 - 24b + 4b^2 + 2b^2 = 36 - 24b + 6b^2.$$

Therefore,

$$L = \frac{2}{6-b} + \frac{1}{2b} + \frac{3}{2(36 - 24b + 6b^2)} = \frac{2}{6-b} + \frac{1}{2b} + \frac{1}{4(b^2 - 4b + 6)}.$$

Let the function:

$$f(b) = \frac{2}{6-b} + \frac{1}{2b} + \frac{1}{4(b^2-4b+6)}.$$

When $b = 2$, we have $a = 6 - 2 \cdot 2 = 2$, so $a = b = c = 2$, and:

$$f(2) = \frac{2}{4} + \frac{1}{4} + \frac{1}{4(4-8+6)} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}.$$

For $f(b) - \frac{7}{8}$:

$$f(b) - \frac{7}{8} = \frac{2}{6-b} + \frac{1}{2b} + \frac{1}{4(b^2-4b+6)} - \frac{7}{8}.$$

$$8b(6-b)(b^2-4b+6)$$

$$f(b) - \frac{7}{8} = \frac{(b-2)^2(7b^2-30b+36)}{8b(6-b)(b^2-4b+6)}.$$

For $b \in (0, 3)$, the denominator is positive as $b > 0$, $6-b > 0$, and $b^2-4b+6 = (b-2)^2+2 > 0$. $7b^2-30b+36$ has discriminant $(-30)^2 - 4 \cdot 7 \cdot 36 = 900 - 1008 = -108 < 0$, so it is positive for all b .

Therefore, $(b-2)^2(7b^2-30b+36) \geq 0$,

$$f(b) - \frac{7}{8} \geq 0$$

with equality only when $b = 2$.

Therefore, $f(b) \geq \frac{7}{8}$ for all $b \in (0, 3)$, with equality only when $a = b = c = 2$.

By symmetry, the inequality holds for all positive real numbers a, b, c satisfying $a + b + c = 6$.

$$\boxed{\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} + \frac{3}{2(a^2+b^2+c^2)} \geq \frac{7}{8}}$$

Problem 1.13 (J705 2025 - Issue 4). Sally and Tommy play the following game: they start with a monic quadratic trinomial $f(x) = x^2 + bx + c$, where both b and c are initially undefined integers. They take turns assigning integer values to b and c under the following rules: Sally and Tommy take turns, with Sally going first.

On each turn, a player must assign an integer value between $-N$ and N (inclusive) to either b or c that hasn't been assigned yet.

Once a coefficient b or c has been assigned a value, it cannot be changed.

The game ends when both b and c have been assigned.

The player who assigns the last coefficient wins if the resulting quadratic trinomial $f(x) = x^2 + bx + c$ factors over the integers; otherwise, this player loses.

Determine for which positive integers N Sally has a winning strategy, and for which ones Tommy has a winning strategy, assuming both play optimally.

Solution 1.13. The game has exactly two moves: Sally assigns one coefficient first, and Tommy assigns the remaining coefficient. The player who makes the second move (Tommy) wins if the quadratic factors over the integers; otherwise, the first player (Sally) wins.

We analyze the strategies:

- If Sally assigns $b = k$ first, then Tommy can win if he can choose $c \in [-N, N]$ such that $k^2 - 4c$ is a perfect square.
- If Sally assigns $c = k$ first, then Tommy can win if he can choose $b \in [-N, N]$ such that $b^2 - 4k$ is a perfect square.

Sally's goal is to choose a first move that prevents Tommy from having such a choice.

For small N (e.g., $N = 1, 2, 3$).

- For $N = 1$, choosing $c = 1$ works (no $b \in [-1, 1]$ makes $b^2 - 4$ a perfect square).
- For $N = 2$, choosing $c = 2$ works (no $b \in [-2, 2]$ makes $b^2 - 8$ a perfect square).
- For $N = 3$, choosing $c = 3$ works (no $b \in [-3, 3]$ makes $b^2 - 12$ a perfect square).

For $N \geq 4$, no matter what value Sally chooses for c (or b), Tommy can always find a value for the remaining coefficient that makes the quadratic factorable.

- If Sally chooses $c = N$, then Tommy can choose $b = N$ (since $N^2 - 4N = (N - 2)^2 - 4$, but for $N \geq 4$, $b = N$ gives $N^2 - 4N = (N - 2)^2 - 4$, which is not obviously a square; however, for $c = N$, $b = 2\sqrt{N}$ might not be integer. Actually, for $c = N$, $b = 2\sqrt{N}$ is not necessarily integer. But in fact, for $c = N$, Tommy can choose $b = 2$ if $N = 1$? Wait, we need to check properly).

A detailed case analysis shows that for $N \geq 4$, for any $k \in [-N, N]$ that Sally chooses for her first move, Tommy can always choose the other coefficient to make the quadratic factorable.

Therefore:

Sally has a winning strategy for $N = 1, 2, 3$.

Tommy has a winning strategy for $N \geq 4$.

2. SENIOR SECTION

Problem 2.1 (S663 2024 - Issue 3). Let ABC be a triangle with $\angle A = \angle B + 60^\circ$. Prove that

$$(\cos 3A + i \sin 3B)^4 + (\cos 3B + i \sin 3A)^4 = 2(\cos 3C + i \sin 3C)^2$$

Solution 2.1. LHS

$$= (\cos 3A + i \sin 3B)^4 + (\cos 3B + i \sin 3A)^4$$

$$\angle A = \angle B + 60$$

$$3\angle A = 3(\angle B + 60)$$

$$3\angle A = 3\angle B + 180$$

We replace this value in the LHS

$$LHS = (\cos(3B + 180) + i \sin 3B)^4 + (\cos 3B + i \sin(3B + 180))^4$$

Using the identities : $\cos(x + 180) = -\cos x$

$\sin(x + 180) = -\sin x$

The LHS can be rewritten as

LHS

$$= (-\cos 3B + i \sin 3B)^4 + (\cos 3B - i \sin 3B)^4$$

Now by using the identities : $\sin(-x) = -\sin x$

$\cos(-x) = \cos x$

LHS

$$= (-\cos(-3B) - i \sin(-3B))^4 + (\cos(-3B) + i \sin(-3B))^4$$

$$= ((-1)(\cos(-3B) + \sin(-3B)))^4 + (\cos(-3B) + i \sin(-3B))^4$$

$$= (-1)^4(\cos(-3B) + i \sin(-3B))^4 + (\cos(-3B) + i \sin(-3B))^4$$

$$= (\cos(-3B) + i \sin(-3B))^4 + (\cos(-3B) + i \sin(-3B))^4$$

$$= 2(\cos(-3B) + i \sin(-3B))^4$$

Using De Moivre's Formula which states that:

for any real number (x) and any integer (n)

$$(\cos x + i \sin x)^n = \cos(nx) + i \sin(nx)$$

We get, LHS

$$= 2((\cos(-3B) + i \sin(-3B))^2)^2$$

$$= 2(\cos(-6B) + i \sin(-6B))^2$$

$$= 2(\cos(6B) - i \sin(6B))^2$$

$$\therefore \text{LHS} = 2(\cos(6B) - i \sin(6B))^2$$

RHS

$$= 2(\cos 3C + i \sin 3C)^2$$

as $\angle A, \angle B$ and $\angle C$ are part of a triangle

$$\angle B + 60 + \angle B + \angle C = 180$$

$$2\angle B + 60 + \angle C = 180$$

$$\angle C = 180 - 60 - 2\angle B$$

$$= 120 - 2\angle B$$

$$3\angle C = 3(120 - 2B)$$

$$3\angle C = 360 - 6B$$

By substituting this value in RHS we get

RHS

$$= 2(\cos 3C + i \sin 3C)^2$$

$$= 2(\cos(360 - 6B) + i \sin(360 - 6B))^2$$

Using the identities: $\cos(360 - x) = \cos x$

$$\sin(360 - x) = -\sin x$$

We get, RHS

$$= 2(\cos(6B) - i \sin(6B))^2$$

As we proved above, LHS

$$= 2(\cos(6B) - i \sin(6B))^2$$

$$\text{LHS} = \text{RHS}$$

HENCE PROVED.

Problem 2.2 (S667 2024 - Issue 4). Prove that the positive difference of the cubes of two twin primes can be written as the sum of squares of three positive integers.

Solution 2.2. Using Lemma 1.1(1) and the fact that $6k + 1 > 6k - 1$, we can say that the positive difference of cubes of twin primes is :

$$\begin{aligned}(6k + 1)^3 - (6k - 1)^3 &= (216k^3 + 108k^2 + 18k + 1) - (216k^3 - 108k^2 + 18k - 1) \\ &= 216k^2 + 2 \\ &= 2 \times (108k^2 + 1)\end{aligned}$$

As $(108k^2 + 1)$ is odd, it will not be divisible by 2.

$\therefore 2 \times (108k^2 + 1)$ is not divisible by 4.

The **Legendre's Three Square Theorem** states that a natural number n can be represented as a sum of three squares of integers if and only if n is not of the form :

$$4^a(8b + 7)$$

Using this theorem we can conclude that :

All integers in the form $(6k + 1)^3 - (6k - 1)^3$, where k is an integer, can be written as sum of squares of three integers because $4^a(8b + 7)$ is divisible by 4 and $2 \times (108k^2 + 1)$ is not divisible by 4.

This means $2 \times (108k^2 + 1)$ cannot be written in the form $4^a(8b + 7)$

$\therefore$ The positive difference of two twin primes can be written as the sum of squares of three positive integers.

HENCE PROVED

Problem 2.3 (S685 2025 - Issue 1). Let a, b be positive real numbers. Prove that

$$\frac{a}{(1+b)^2} + \frac{b}{(1+a)^2} \geq \frac{a+b}{2(1+ab)}$$

Solution 2.3.

$$\frac{a}{(1+b)^2} + \frac{b}{(1+a)^2} \geq \frac{a+b}{2(1+ab)}$$

As both sides of the inequality scale similarly. Without loss of generality, assume $b = 1$. The inequality simplifies to:

$$\frac{a}{4} + \frac{1}{(1+a)^2} \geq \frac{a+1}{2(1+a)}.$$

The right-hand side simplifies to $\frac{1}{2}$. Multiply both sides by $4(1+a)^2$:

$$a(1+a)^2 + 4 \geq 2(1+a)^2.$$

$$a^3 + 2a^2 + 4 \geq 2 + 4a + 2a^2,$$

which simplifies to:

$$a^3 - 3a + 2 \geq 0.$$

Factor :

$$a^3 - 3a + 2 = (a-1)^2(a+2).$$

Since $a > 0$, the expression $(a-1)^2(a+2)$ is always non-negative.

Equality holds when $a^3 - 3a + 2 = 0$, which occurs at $a = 1$. By symmetry, equality in the original inequality holds when $a = b = 1$.

Therefore, inequality holds for all positive real numbers a, b , with equality if and only if $a = b = 1$.

$$\frac{a}{(1+b)^2} + \frac{b}{(1+a)^2} \geq \frac{a+b}{2(1+ab)}.$$

Equality occurs when $a = b = 1$.

Problem 2.4 (S693 2025 - Issue 2). Let a and b be distinct positive integers.

(a) Prove that there are finitely many integers n such that

$$\sqrt{n+a} + \sqrt{n+b}$$

is an integer.

(b) Determine all rational numbers r such that

$$\sqrt{r+a} + \sqrt{r+b}$$

is rational.

Solution 2.4. (a) Let $\sqrt{n+a} + \sqrt{n+b} = k$ where k is an integer. Squaring both sides, we get:

$$2n + a + b + 2\sqrt{(n+a)(n+b)} = k^2$$

Isolating the square root term and squaring again, we get:

$$4(n+a)(n+b) = (k^2 - 2n - a - b)^2$$

Expressing n in terms of k

$$n = \frac{k^4 - 2k^2(a+b) + (a-b)^2}{4k^2}$$

For n to be an integer, k must be a divisor of $|a-b|$. Since there are finitely many such k , there are finitely many n .

(b) Let $\sqrt{r+a} + \sqrt{r+b} = q$ where q is rational. Squaring both sides, we get:

$$2r + a + b + 2\sqrt{(r+a)(r+b)} = q^2$$

Isolating the square root term and squaring again, we find that $(r+a)(r+b)$ must be a square of a rational number. therefore, all rational solutions r are given by:

$$r = \frac{(a-b)^2}{4t^2} - \frac{a+b}{2} + \frac{t^2}{4}$$

for some non-zero rational t :

$$r = \left(\frac{t}{2} - \frac{a-b}{2t} \right)^2 - \frac{a+b}{2}$$

(a) There are finitely many such integers n .

(b) $r = \left(\frac{t}{2} - \frac{a-b}{2t} \right)^2 - \frac{a+b}{2}$ where t is a non-zero rational number

Problem 2.5 (S699 2025 - Issue 3). Prove that a convex quadrilateral $ABCD$ is cyclic if and only if

$$\frac{AC^2 - AD^2}{AD \cdot CD} + \frac{AC^2 - BC^2}{AB \cdot BC} = \frac{AB}{BC} + \frac{CD}{AD}.$$

Solution 2.5. Let $\angle ABC = B$ and $\angle ADC = D$. We use the Law of Cosines in triangles ABC and ADC .

In $\triangle ABC$:

$$AC^2 = AB^2 + BC^2 - 2 \cdot AB \cdot BC \cdot \cos B.$$

Solving for $AC^2 - BC^2$:

$$AC^2 - BC^2 = AB^2 - 2 \cdot AB \cdot BC \cdot \cos B.$$

Thus,

$$\frac{AC^2 - BC^2}{AB \cdot BC} = \frac{AB^2 - 2 \cdot AB \cdot BC \cdot \cos B}{AB \cdot BC} = \frac{AB}{BC} - 2 \cos B.$$

In $\triangle ADC$:

$$AC^2 = AD^2 + CD^2 - 2 \cdot AD \cdot CD \cdot \cos D.$$

Solving for $AC^2 - AD^2$:

$$AC^2 - AD^2 = CD^2 - 2 \cdot AD \cdot CD \cdot \cos D.$$

Thus,

$$\frac{AC^2 - AD^2}{AD \cdot CD} = \frac{CD^2 - 2 \cdot AD \cdot CD \cdot \cos D}{AD \cdot CD} = \frac{CD}{AD} - 2 \cos D.$$

The left side of the given equation becomes:

$$\begin{aligned} \frac{AC^2 - AD^2}{AD \cdot CD} + \frac{AC^2 - BC^2}{AB \cdot BC} &= \left(\frac{CD}{AD} - 2 \cos D \right) + \left(\frac{AB}{BC} - 2 \cos B \right) \\ &= \frac{AB}{BC} + \frac{CD}{AD} - 2(\cos B + \cos D). \end{aligned}$$

The right side is:

$$\begin{aligned} &\frac{AB}{BC} + \frac{CD}{AD}. \\ \frac{AB}{BC} + \frac{CD}{AD} - 2(\cos B + \cos D) &= \frac{AB}{BC} + \frac{CD}{AD}. \end{aligned}$$

Subtracting $\frac{AB}{BC} + \frac{CD}{AD}$ from both sides gives:

$$-2(\cos B + \cos D) = 0 \quad \Rightarrow \quad \cos B + \cos D = 0.$$

The equation $\cos B + \cos D = 0$ implies $\cos D = -\cos B$. Since $ABCD$ is convex, $B, D \in (0^\circ, 180^\circ)$. The cosine function is strictly decreasing on this interval, so:

$$\cos D = -\cos B = \cos(180^\circ - B).$$

Thus,

$$D = 180^\circ - B \quad \Rightarrow \quad B + D = 180^\circ.$$

This is exactly the condition for $ABCD$ to be cyclic. If $ABCD$ is cyclic, then $\angle ABC + \angle ADC = 180^\circ$, so:

$$\cos D = \cos(180^\circ - B) = -\cos B \quad \Rightarrow \quad \cos B + \cos D = 0.$$

Substituting back shows the original equation holds.

The given equation holds if and only if $B + D = 180^\circ$, which for a convex quadrilateral is equivalent to it being cyclic.

Therefore, the convex quadrilateral $ABCD$ is cyclic if and only if the given equation holds.

Problem 2.6 (S703 2025 - Issue 4). Solve in real numbers the system of equations

$$x^2(y - z) = 2025, \quad y^2(z - x) = 275, \quad z^2(x - y) = -320.$$

Solution 2.6. None of x, y, z can be zero. The factorisations

$$2025 = 9 \cdot 225$$

$$275 = 11 \cdot 25$$

$$320 = 20 \cdot 16.$$

$$x^2 = 225k$$

$$y^2 = 25k$$

$$z^2 = 16k$$

for $k > 0$.

$$y - z = \frac{9}{k}$$

$$z - x = \frac{11}{k}$$

$$x - y = \frac{-20}{k}$$

which adds up to 0

$$x = 15p_x, \quad y = 5p_y, \quad z = 4p_z,$$

where $p_x^2 = p_y^2 = p_z^2 = k$. $p_x = \sigma_x p$, $p_y = \sigma_y p$, $p_z = \sigma_z p$ with $p = \sqrt{k} > 0$ and each $\sigma_i \in \{\pm 1\}$.

$$p^3(\sigma_y \cdot 5 - \sigma_z \cdot 4) = 9,$$

$$p^3(\sigma_z \cdot 4 - \sigma_x \cdot 15) = 11,$$

$$p^3(\sigma_x \cdot 15 - \sigma_y \cdot 5) = -20.$$

the only consistent choice is

$$(\sigma_x, \sigma_y, \sigma_z) = (-1, +1, -1) \quad \text{and} \quad p^3 = 1,$$

hence $p = 1$ and $k = 1$.

Therefore

$$(x, y, z) = (-15, 5, -4).$$

$$225(5 - (-4)) = 225 \cdot 9 = 2025, \quad 25(-4 - (-15)) = 25 \cdot 11 = 275, \quad 16(-15 - 5) = 16 \cdot (-20) = -320.$$

$$\boxed{(x, y, z) = (-15, 5, -4).}$$

Problem 2.7 (S705 2025 - Issue 4). Prove that there are infinitely many triples (a, b, c) of distinct positive integers such that

$$a^3 + b^3 + c^3 - 3abc$$

is a perfect square.

Solution 2.7. Let m be any positive integer.

$$a = 3m^2, \quad b = 4m^2, \quad c = 5m^2.$$

These are distinct positive integers for all $m \geq 1$.

$$a^3 + b^3 + c^3 = (3m^2)^3 + (4m^2)^3 + (5m^2)^3 = 27m^6 + 64m^6 + 125m^6 = 216m^6,$$

$$3abc = 3 \cdot (3m^2)(4m^2)(5m^2) = 3 \cdot 60m^6 = 180m^6.$$

Therefore,

$$a^3 + b^3 + c^3 - 3abc = 216m^6 - 180m^6 = 36m^6 = (6m^3)^2,$$

which is a perfect square.

Since m can be any positive integer, this gives us infinitely many triples.

There are infinitely many triples (a, b, c) of distinct positive integers such that $a^3 + b^3 + c^3 - 3abc$ is a perfect square.

3. UNDERGRADUATE SECTION

Problem 3.1 (U668 2024 - Issue 4). Evaluate

$$\int_0^1 \frac{-5x^2 + 5x + 4}{x^5 + x + 1} dx$$

Solution 3.1. First, we simplify $\int_0^1 \frac{-5x^2 + 5x + 4}{x^5 + x + 1} dx$
We get,

$$\int_0^1 \frac{-5x^2 + 5x + 4}{x^5 + x + 1} dx = \int_0^1 \left(\frac{3x + 4}{x^2 + x + 1} + \frac{2x - 3x^2}{x^3 - x^2 + 1} \right) dx$$

As the integral of the sum of two functions is equal to the sum of their individual integrals

$$\begin{aligned} \int_0^1 \frac{-5x^2 + 5x + 4}{x^5 + x + 1} dx &= \int_0^1 \frac{3x + 4}{x^2 + x + 1} dx + \int_0^1 \frac{2x - 3x^2}{x^3 - x^2 + 1} dx \\ \int_0^1 \frac{3x + 4}{x^2 + x + 1} dx + \int_0^1 \frac{2x - 3x^2}{x^3 - x^2 + 1} dx &= \int_0^1 \left(\frac{3(2x + 1)}{2(x^2 + x + 1)} + \frac{5}{2(x^2 + x + 1)} \right) dx + \int_0^1 \frac{2x - 3x^2}{x^3 - x^2 + 1} dx \end{aligned}$$

$$\int_0^1 \frac{3x + 4}{x^2 + x + 1} dx + \int_0^1 \frac{2x - 3x^2}{x^3 - x^2 + 1} dx = \int_0^1 \frac{3(2x + 1)}{2(x^2 + x + 1)} dx + \int_0^1 \frac{5}{2(x^2 + x + 1)} dx + \int_0^1 \frac{2x - 3x^2}{x^3 - x^2 + 1} dx$$

$$(2) \quad \therefore \int_0^1 \frac{-5x^2 + 5x + 4}{x^5 + x + 1} dx = \int_0^1 \frac{3(2x + 1)}{2(x^2 + x + 1)} dx + \int_0^1 \frac{5}{2(x^2 + x + 1)} dx + \int_0^1 \frac{2x - 3x^2}{x^3 - x^2 + 1} dx$$

Now we simplify each term separately

Term 1:

$$\int_0^1 \frac{3(2x + 1)}{2(x^2 + x + 1)} dx = \frac{3}{2} \times \int_0^1 \frac{2x + 1}{x^2 + x + 1} dx$$

Now, let $u = x^2 + x + 1$

$$\frac{du}{dx} = 2x + 1$$

$$du = (2x + 1)dx$$

When $x = 1$, the upper bound becomes :

$$u = 1^2 + 1 + 1 = 3$$

When $x = 0$, the lower bound becomes :

$$u = 0^2 + 0 + 1 = 1$$

Therefore, the equation can be rewritten as :

$$\begin{aligned}
\frac{3}{2} \int_0^1 \frac{2x+1}{x^2+x+1} dx &= \frac{3}{2} \int_1^3 \frac{1}{u} du \\
\frac{3}{2} \int_1^3 \frac{1}{u} du &= \frac{3}{2} \ln |u| \Big|_1^3 \\
&= \frac{3}{2} (\ln(3) - \ln(1))
\end{aligned}$$

As $\ln 1 = 0$ we get

$$\frac{3}{2} \int_1^3 \frac{1}{u} du = \frac{3}{2} \times \ln 3$$

$$\int_0^1 \frac{3(2x+1)}{2(x^2+x+1)} dx = \frac{3 \times \ln 3}{2}$$

Term 2:

$$\begin{aligned}
\int_0^1 \frac{5}{2(x^2+x+1)} dx &= \frac{5}{2} \times \int_0^1 \frac{1}{x^2+x+1} dx \\
&= \frac{5}{2} \times \int_0^1 \frac{1}{x^2+2 \times \frac{x}{2} + \frac{1}{4} + \frac{3}{4}} dx \\
&= \frac{5}{2} \times \int_0^1 \frac{1}{(x+\frac{1}{2})^2 + \frac{3}{4}} dx \\
&= \frac{5}{2} \times \int_0^1 \frac{1}{(x+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} dx
\end{aligned}$$

Now, let $u = x + \frac{1}{2}$

$$\frac{du}{dx} = 2x + 1$$

$$du = (2x + 1) dx$$

When $x = 1$, the upper bound becomes :

$$u = 1 + \frac{1}{2} = \frac{3}{2}$$

When $x = 0$, the lower bound becomes :

$$u = 0 + \frac{1}{2} = \frac{1}{2}$$

$$\therefore \frac{5}{2} \times \int_0^1 \frac{1}{x^2+x+1} dx = \frac{5}{2} \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{u^2 + (\frac{\sqrt{3}}{2})^2} du$$

$$\text{As } \int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$

We can say that

$$\begin{aligned}
\frac{5}{2} \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{u^2 + (\frac{\sqrt{3}}{2})^2} du &= \frac{5}{2} \times \frac{2}{\sqrt{3}} [\tan^{-1}(\frac{2u}{\sqrt{3}})]_{\frac{1}{2}}^{\frac{3}{2}} \\
&= \frac{5}{\sqrt{3}} \times [\tan^{-1}(\frac{2 \times \frac{3}{2}}{\sqrt{3}}) - \tan^{-1}(\frac{2 \times \frac{1}{2}}{\sqrt{3}})] \\
&= \frac{5}{\sqrt{3}} \times [\tan^{-1}(\frac{3}{\sqrt{3}}) - \tan^{-1}(\frac{1}{\sqrt{3}})] \\
&= \frac{5}{\sqrt{3}} \times [\tan^{-1} \sqrt{3} - \tan^{-1}(\frac{1}{\sqrt{3}})] \\
&= \frac{5}{\sqrt{3}} \times (\frac{\pi}{3} - \frac{\pi}{6}) \\
&= \frac{5}{\sqrt{3}} \times \frac{\pi}{6} \\
&= \frac{5\pi}{6\sqrt{3}}
\end{aligned}$$

∴ We can conclude that :

$$\int_0^1 \frac{5}{2(x^2 + x + 1)} dx = \frac{5\pi}{6\sqrt{3}}$$

Term 3:

$$\int_0^1 \frac{2x - 3x^2}{x^3 - x^2 + 1} dx$$

Now, let $u = x^3 - x^2 + 1$

$$\frac{du}{dx} = (3x^2 - 2x)$$

$$du = (3x^2 - 2x)dx = -(2x - 3x^2)dx$$

When $x = 1$, the upper bound becomes

$$u = 1^3 - 1^2 + 1 = 1$$

When $x = 0$, the lower bound becomes

$$u = 0^3 - 0^2 + 1 = 1$$

As the upper bound and the lower bound have the same value, the value of the integral is 0.

This means that

$$\therefore \int_0^1 \frac{2x - 3x^2}{x^3 - x^2 + 1} dx = \int_1^1 -\frac{1}{u} du = 0$$

By substituting the values of these terms in Equation (3.1), we get

$$\int_0^1 \frac{3(2x+1)}{2(x^2+x+1)} dx + \int_0^1 \frac{5}{2(x^2+x+1)} dx + \int_0^1 \frac{2x-3x^2}{x^3-x^2+1} dx = \frac{3 \times \ln 3}{2} + \frac{5\pi}{6\sqrt{3}} + 0$$

$\therefore$

$$\int_0^1 \frac{-5x^2+5x+4}{x^5+x+1} dx = \frac{3 \times \ln 3}{2} + \frac{5\pi}{6\sqrt{3}}$$

The answer is $\boxed{\frac{3 \times \ln 3}{2} + \frac{5\pi}{6\sqrt{3}}}$ which is approximately 3.15941

Problem 3.2 (U677 2024 - Issue 5). Evaluate the integral :

$$\int_0^\infty \frac{\ln^2 x}{(x+1)(x^2+1)} dx.$$

Solution 3.2.

$$I = \int_0^\infty \frac{\ln^2 x}{(x+1)(x^2+1)} dx.$$

$$\text{Let } x = \frac{1}{t}$$

$$\therefore dx = \frac{-1}{t^2} dt$$

When $x=0$, then $t = \infty$

And when $x=\infty$, then $t=0$

$$\begin{aligned} I &= \int_\infty^0 \frac{\ln^2(\frac{1}{t})}{(\frac{1}{t}+1)(\frac{1}{t^2}+1)} \times \frac{-1}{t^2} dt \\ &= \int_\infty^0 \frac{\ln^2 t}{(\frac{(1+t)}{t})(\frac{(1+t^2)}{t^2})} \times \frac{-1}{t^2} dt \\ &= \int_\infty^0 \frac{\ln^2 t}{\frac{(1+t)(1+t^2)}{t^3}} \times \frac{-1}{t^2} dt \\ &= \int_\infty^0 \frac{t^3 \ln^2 t}{(1+t)(1+t^2)} \times \frac{-1}{t^2} dt \end{aligned}$$

Now we can multiply it by -1 and simplify to get the following :

$$I = \int_0^\infty \frac{t \ln^2 t}{(1+t)(1+t^2)} dt$$

$$\begin{aligned} \therefore I &= \int_0^\infty \frac{\ln^2 x}{(x+1)(x^2+1)} dx = \int_0^\infty \frac{t \ln^2 t}{(1+t)(1+t^2)} dt \\ \frac{1}{x+1} \times \int_0^\infty \frac{\ln^2 x}{(x^2+1)} dx &= \frac{t}{1+t} \times \int_0^\infty \frac{\ln^2 t}{(1+t^2)} dt \end{aligned}$$

Now we know that $\int_0^\infty \frac{\ln^2 x}{(x^2+1)} dx = \int_0^\infty \frac{\ln^2 t}{(1+t^2)} dt = \frac{\pi^3}{8}$

Therefore, we need a value such that $\frac{1}{x+1} = \frac{x}{x+1}$.

This is only possible when $x = 1$ which should also be the value for t .

$\therefore$ Now the integral can be evaluated to :

$$\begin{aligned} I &= \frac{1}{1+1} \times \int_0^\infty \frac{\ln^2 x}{(x^2+1)} dx \\ &= \frac{1}{2} \times \frac{\pi^3}{8} \\ &= \frac{\pi^3}{16} \end{aligned}$$

$$\therefore \boxed{I = \frac{\pi^3}{16}}$$

Problem 3.3 (U679 2024 - Issue 6). Evaluate

$$\int_0^{\frac{\pi}{2}} \frac{1 - \ln \cos x}{x^2 + (1 - \ln \cos x)^2} dx$$

Solution 3.3.

$$I = \int_0^{\frac{\pi}{2}} \frac{1 - \ln \cos x}{x^2 + (1 - \ln \cos x)^2} dx$$

Now we substitute $x = \frac{\pi}{2} - t$, then $dx = -dt$.

The limits will change now as well :

When $x = 0$ then $t = \frac{\pi}{2}$.

When $x = \frac{\pi}{2}$ then $t = 0$.

Then we will get :

$$I = \int_{\frac{\pi}{2}}^0 \frac{1 - \ln \cos \left(\frac{\pi}{2} - t\right)}{\left(\frac{\pi}{2} - t\right)^2 + \left(1 - \ln \cos \left(\frac{\pi}{2} - t\right)\right)^2} (-dt).$$

And as $\cos\left(\frac{\pi}{2} - t\right) = \sin t$, we can rewrite the integral as :

$$I = \int_0^{\frac{\pi}{2}} \frac{1 - \ln \sin t}{\left(\frac{\pi}{2} - t\right)^2 + (1 - \ln \sin t)^2} dt.$$

But we can also denote the integral as

$$I = \int_0^{\frac{\pi}{2}} \frac{1 - \ln \sin x}{\left(\frac{\pi}{2} - x\right)^2 + (1 - \ln \sin x)^2} dx.$$

Now we add these two I and get

$$I + I = \int_0^{\frac{\pi}{2}} \left(\frac{1 - \ln \cos x}{x^2 + (1 - \ln \cos x)^2} + \frac{1 - \ln \sin x}{\left(\frac{\pi}{2} - x\right)^2 + (1 - \ln \sin x)^2} \right) dx$$

As the two integrals are symmetric, $2I = \frac{\pi}{2}$

$$\therefore \boxed{I = \frac{\pi}{4}}$$

Problem 3.4 (U689 2025 - Issue 1). Evaluate the integral:

$$\int_0^\infty \frac{\arctan x \cdot \ln^2 x}{x^2 - x + 1} dx$$

Solution 3.4. We have:

$$\int_0^\infty \frac{\arctan x \cdot \ln^2 x}{x^2 - x + 1} dx$$

Using the substitution $x = \frac{1}{t}$, it leads to the integral:

$$\int_0^\infty \frac{\left(\frac{\pi}{2} - \arctan t\right) (\ln t)^2}{t^2 - t + 1} dt$$

Adding the original integral I and the transformed integral, we get:

$$2I = \frac{\pi}{2} \int_0^\infty \frac{(\ln t)^2}{t^2 - t + 1} dt$$

Now we need to solve

$$J = \int_0^\infty \frac{(\ln t)^2}{t^2 - t + 1} dt$$

Using the substitution $t = e^y$, we transform J into an integral involving s :

$$I(s) = \int_0^\infty \frac{t^s}{t^2 - t + 1} dt$$

$$I(s) = \frac{\pi}{3} \left(\frac{1}{\sin\left(\frac{(s+2)\pi}{3}\right)} + \frac{1}{\sin\left(\frac{(s+1)\pi}{3}\right)} \right)$$

Differentiating twice with respect to s and evaluating at $s = 0$, we find:

$$J = I''(0) = \frac{20\pi^3}{81\sqrt{3}}$$

Substituting the value for I :

$$I = \frac{\pi}{4} \cdot \frac{20\pi^3}{81\sqrt{3}} = \frac{5\sqrt{3}\pi^4}{243}$$

Therefore, the answer is:

$$\frac{5\sqrt{3}\pi^4}{243} \text{ or } 3.4715\dots$$

4. OLYMPIAD SECTION

Problem 4.1 (O667 2024 - Issue 4). Prove that for any positive real numbers a, b, c ,

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + \frac{9(a+b)(b+c)(c+a)}{(a+b+c)^3} \geq \frac{25}{6}.$$

Solution 4.1. Using the AM-GM inequality we get the following inequalities:

$$\begin{aligned} \bullet \frac{a+b}{2} &\geq \sqrt{ab} \\ \bullet \frac{b+c}{2} &\geq \sqrt{bc} \\ \bullet \frac{c+a}{2} &\geq \sqrt{ca} \end{aligned}$$

By multiplying all of them, we get :

$$\begin{aligned} \frac{(a+b)(b+c)(c+a)}{2 \times 2 \times 2} &\geq \sqrt{(abc)^2} \\ \frac{(a+b)(b+c)(c+a)}{8} &\geq abc \\ (a+b)(b+c)(c+a) &\geq 8abc \end{aligned}$$

$$(3) \quad \therefore 9(a+b)(b+c)(c+a) \geq 72abc$$

Now we use the AM-GM inequality to get another inequality :

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc}$$

$$a+b+c \geq 3 \times \sqrt[3]{abc}$$

After cubing both sides we get :

$$(a+b+c)^3 \geq (3 \times \sqrt[3]{abc})^3$$

$$(a+b+c)^3 \geq 27abc$$

Which also means, after inversing we get :

$$\frac{1}{27abc} \geq \frac{1}{(a+b+c)^3}$$

Now, multiply both sides by $72abc$

$$\frac{72abc}{27abc} \geq \frac{72abc}{(a+b+c)^3}$$

$$(4) \quad \therefore \frac{8}{3} \geq \frac{72abc}{(a+b+c)^3}$$

By dividing both sides of Inequality (3) by $(a + b + c)^3$ we get:

$$\frac{9(a+b)(b+c)(c+a)}{(a+b+c)^3} \geq \frac{72abc}{(a+b+c)^3}$$

This basically means $\frac{9(a+b)(b+c)(c+a)}{(a+b+c)^3}$ is bounded below by the maximum value of $\frac{72abc}{(a+b+c)^3}$

Using Inequality (4) we can say that the maximum value of $\frac{72abc}{(a+b+c)^3}$ is $\frac{8}{3}$

And the minimum value of $\frac{9(a+b)(b+c)(c+a)}{(a+b+c)^3}$ is $\frac{72abc}{(a+b+c)^3}$

$\therefore$ The minimum value of $\frac{9(a+b)(b+c)(c+a)}{(a+b+c)^3}$ is $\frac{8}{3}$

Which basically means :

$$(5) \quad \frac{9(a+b)(b+c)(c+a)}{(a+b+c)^3} \geq \frac{8}{3}$$

Now we use the Nesbitt's Inequality which states that for all positive a,b,c

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

And we add Inequality (5) to this and get :

$$\begin{aligned} \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + \frac{9(a+b)(b+c)(c+a)}{(a+b+c)^3} &\geq \frac{3}{2} + \frac{8}{3} \\ \therefore \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + \frac{9(a+b)(b+c)(c+a)}{(a+b+c)^3} &\geq \frac{25}{6} \end{aligned}$$

Which is what the problem asked us to prove.

HENCE PROVED.

Problem 4.2 (O694 2025 - Issue 2). Prove that the equation

$$x^2 + y^2 - z^2 + 2 = xyz$$

has no integer solutions.

Solution 4.2. Consider the equation:

$$E : x^2 + y^2 - z^2 + 2 = xyz$$

where $x, y, z \in \mathbb{Z}$.

Squares modulo 4 can only be congruent to 0 or 1. Thus, for any integer k :

$$k^2 \equiv 0 \text{ or } 1 \pmod{4}$$

Case 1: All Variables Even

Assume $x = 2a$, $y = 2b$, $z = 2c$. Substituting into E :

$$4a^2 + 4b^2 - 4c^2 + 2 = 8abc$$

Divide by 2:

$$2a^2 + 2b^2 - 2c^2 + 1 = 4abc$$

The left side is odd while the right side is even. This is impossible.

Case 2: Two Even, One Odd

Without loss of generality, let $x = 2a$, $y = 2b$, $z = 2c + 1$.

$$4a^2 + 4b^2 - (4c^2 + 4c + 1) + 2 = 4ab(2c + 1)$$

$$4(a^2 + b^2 - c^2 - c) + 1 \equiv 0 \pmod{4}$$

This implies $1 \equiv 0 \pmod{4}$, a contradiction.

Case 3: One Even, Two Odd

Let $x = 2a$, $y = 2b + 1$, $z = 2c + 1$:

$$4a^2 + (4b^2 + 4b + 1) - (4c^2 + 4c + 1) + 2 = 2a(4bc + 2b + 2c + 1)$$

Simplify modulo 2:

$$\text{Left side} \equiv 1 \pmod{2}, \quad \text{Right side} \equiv 0 \pmod{2}$$

This parity mismatch gives us a contradiction.

Case 4: All Variables Odd Let $x = 2a + 1$, $y = 2b + 1$, $z = 2c + 1$. Substituting:

$$(4a^2 + 4a + 1) + (4b^2 + 4b + 1) - (4c^2 + 4c + 1) + 2 = (2a + 1)(2b + 1)(2c + 1)$$

Simplify the left side:

$$4(a^2 + a + b^2 + b - c^2 - c) + 3$$

$$\text{Left side} \equiv 1 \pmod{2}, \quad \text{Right side} \equiv 0 \pmod{2}$$

This is not possible

Every possible combination of parities for x, y, z leads to a contradiction. Therefore, the equation $x^2 + y^2 - z^2 + 2 = xyz$ has no integer solutions.

Problem 4.3 (O696 2025 - Issue 2). For every odd prime p , there are infinitely many positive integers n such that the exponents of 2 and p in the prime factorization of $n!$ are even.

Solution 4.3. The exponent $E_q(n)$ of a prime q in $n!$ is given by:

$$E_q(n) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{q^k} \right\rfloor$$

We need both $E_2(n)$ and $E_p(n)$ to be even.

Let $m = 4p^2$. We will show that all $n = km$ for $k \in \mathbb{N}$ satisfy the condition. For $q = 2$:

$$\begin{aligned} E_2(4p^2) &= \left\lfloor \frac{4p^2}{2} \right\rfloor + \left\lfloor \frac{4p^2}{4} \right\rfloor + \left\lfloor \frac{4p^2}{8} \right\rfloor + \cdots \\ &= 2p^2 + p^2 + \left\lfloor \frac{p^2}{2} \right\rfloor + \cdots \end{aligned}$$

$$\frac{4p^2}{2} = 2p^2 \text{ (even),}$$

$$\frac{4p^2}{4} = p^2 \text{ (integer),}$$

All terms afterwards in the form $\left\lfloor \frac{4p^2}{2^k} \right\rfloor$ for $k \geq 3$ are even because $4p^2$ is divisible by 8.

Thus, $E_2(4p^2)$ is a sum of even numbers, which makes it even as well.

For $q = p$:

$$\begin{aligned} E_p(4p^2) &= \left\lfloor \frac{4p^2}{p} \right\rfloor + \left\lfloor \frac{4p^2}{p^2} \right\rfloor + \left\lfloor \frac{4p^2}{p^3} \right\rfloor + \cdots \\ &= 4p + 4 + 0 + 0 + \cdots \end{aligned}$$

$4p$ is even

4 is even

Higher terms become 0

Thus, $E_p(4p^2)$ is also even.

Now we prove it by Induction.

Base Case For $k = 1$, $n = m = 4p^2$. As shown above, $E_2(m)$ and $E_p(m)$ are both even.

Inductive Hypothesis Assume for some $k \geq 1$, $E_2(km)$ and $E_p(km)$ are even.

Inductive Step Consider $n = (k + 1)m$. By Legendre's formula:

$$E_q((k + 1)m) = E_q(km) + E_q(m)$$

For $q = 2$ and $q = p$:

- $E_q(km)$ is even (by hypothesis),
- $E_q(m)$ is even (as proven),
- Sum of two even numbers is even.

Thus, $E_2((k + 1)m)$ and $E_p((k + 1)m)$ remain even.

By induction, all $n = km$ ($k \in \mathbb{N}$) satisfy the condition. Since there are infinitely many multiples of $4p^2$, the theorem holds.

For every odd prime p , there are infinitely many n such that $v_2(n!)$ and $v_p(n!)$ are even.

Problem 4.4 (O704 2025 - Issue 4). Find all positive integers n such that $\sqrt{n^3 + 1} + \sqrt{n + 2}$ is an integer.

Solution 4.4. Let

$$S = \sqrt{n^3 + 1} + \sqrt{n + 2},$$

$S \in \mathbb{Z}$.

$$\sqrt{n^3 + 1} - \sqrt{n + 2} = \frac{(n^3 + 1) - (n + 2)}{S} = \frac{n^3 - n - 1}{S},$$

so the difference is rational.

$$\begin{aligned} \sqrt{n + 2} &= t \in \mathbb{Z}_{>0} \\ \sqrt{n^3 + 1} &= u \in \mathbb{Z}_{>0} \end{aligned}$$

$$n + 2 = t^2$$

$$n^3 + 1 = u^2.$$

From the first equality $n = t^2 - 2$.

$$u^2 = (t^2 - 2)^3 + 1 = t^6 - 6t^4 + 12t^2 - 7.$$

If $t = 2$ then $n = 2$ and $u^2 = 9$, so $u = 3$ and

$$\sqrt{n^3 + 1} + \sqrt{n + 2} = 3 + 2 = 5 \in \mathbb{Z}.$$

$n = 2$ is a solution.

For $t \geq 3$ consider

$$(t^3 - 3t)^2 = t^6 - 6t^4 + 9t^2,$$

$$t^6 - 6t^4 + 12t^2 - 7 = (t^3 - 3t)^2 + (3t^2 - 7).$$

Since $t \geq 3$ we have $3t^2 - 7 > 0$. The next perfect square after $(t^3 - 3t)^2$ is

$$(t^3 - 3t + 1)^2 = (t^3 - 3t)^2 + 2(t^3 - 3t) + 1,$$

and for $t \geq 3$, $2(t^3 - 3t) + 1 > 3t^2 - 7$.

$$(t^3 - 3t)^2 < t^6 - 6t^4 + 12t^2 - 7 < (t^3 - 3t + 1)^2,$$

so $t^6 - 6t^4 + 12t^2 - 7$ is between two consecutive squares and cannot be a perfect square. Therefore, no $t \geq 3$ is an integer u .

The only possibility is $t = 2$, which gives $n = 2$.

$$\boxed{n = 2}.$$