

MATHEMATICAL REFLECTIONS JOURNAL JULY & SEPTEMBER'24

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1. UNDERGRADUATE MATHEMATICS

Problem 1.1 (U 664, July, Issue 3, 2024). For every integer $n \geq 0$ define

$$I_n = \int \frac{x^n e^{\arctan x}}{\sqrt{1+x^2}} dx.$$

Prove that

$$(n+1)I_{n+1} + I_n + nI_{n-1} = x^n \sqrt{1+x^2} e^{\arctan x}$$

for all $n > 0$ and then evaluate I_2 .**Solution 1.1** (U664). Given:

$$f(x) = x^n \sqrt{1+x^2} e^{\arctan x}$$

The derivative of $f(x)$ is:

$$f'(x) = (x^n)' \sqrt{1+x^2} e^{\arctan x} + (\sqrt{1+x^2})' x^n e^{\arctan x} + (e^{\arctan x})' \sqrt{1+x^2} x^n$$

Now, let's differentiate each part individually:

1. Differentiating x^n :

$$(x^n)' = nx^{n-1}$$

2. Differentiating $\sqrt{1+x^2}$:

$$(\sqrt{1+x^2})' = \frac{1}{2}(1+x^2)^{-1/2} \cdot 2x = \frac{x}{\sqrt{1+x^2}}$$

3. Differentiating $e^{\arctan x}$:

$$(e^{\arctan x})' = e^{\arctan x} \cdot \frac{1}{1+x^2}$$

Substitute these derivatives back into the product rule formula:

$$f'(x) = nx^{n-1} \sqrt{1+x^2} e^{\arctan x} + \frac{x}{\sqrt{1+x^2}} x^n e^{\arctan x} + e^{\arctan x} \cdot \frac{1}{1+x^2} \sqrt{1+x^2} x^n$$

Simplify each term:

1. Simplifying $nx^{n-1} \sqrt{1+x^2} e^{\arctan x}$:

$$nx^{n-1} \sqrt{1+x^2} e^{\arctan x} = \frac{nx^{n-1}(1+x^2)e^{\arctan x}}{\sqrt{1+x^2}} = \frac{nx^{n-1}e^{\arctan x}}{\sqrt{1+x^2}} + \frac{nx^{n+1}e^{\arctan x}}{\sqrt{1+x^2}}$$

2. Simplifying $\frac{x}{\sqrt{1+x^2}} x^n e^{\arctan x}$:

$$\frac{x}{\sqrt{1+x^2}} x^n e^{\arctan x} = \frac{x^{n+1} e^{\arctan x}}{\sqrt{1+x^2}}$$

3. Simplifying $e^{\arctan x} \cdot \frac{1}{1+x^2} \sqrt{1+x^2} x^n$:

$$e^{\arctan x} \cdot \frac{1}{1+x^2} \sqrt{1+x^2} x^n = \frac{x^n e^{\arctan x}}{\sqrt{1+x^2}}$$

Combining these terms, we get:

$$f'(x) = \frac{nx^{n-1}e^{\arctan x}}{\sqrt{1+x^2}} + \frac{nx^{n+1}e^{\arctan x}}{\sqrt{1+x^2}} + \frac{x^{n+1}e^{\arctan x}}{\sqrt{1+x^2}} + \frac{x^n e^{\arctan x}}{\sqrt{1+x^2}}$$

Combine like terms:

$$f'(x) = \frac{(n+1)x^{n+1}e^{\arctan x}}{\sqrt{1+x^2}} + \frac{x^n e^{\arctan x}}{\sqrt{1+x^2}} + \frac{nx^{n-1}e^{\arctan x}}{\sqrt{1+x^2}}$$

Thus, we have:

$$(x^n \sqrt{1+x^2} e^{\arctan x})' = \frac{(n+1)x^{n+1}e^{\arctan x}}{\sqrt{1+x^2}} + \frac{x^n e^{\arctan x}}{\sqrt{1+x^2}} + \frac{nx^{n-1}e^{\arctan x}}{\sqrt{1+x^2}}$$

Integrating both sides, we get:

$$x^n \sqrt{1+x^2} e^{\arctan x} = (n+1)I_{n+1} + I_n + nI_{n-1}$$

For finding I_2 , we put $n = 1$ in the proven identity, we get $2I_2 + (I_0 + I_1) = x\sqrt{1+x^2}e^{\arctan x}$. We will first evaluate $I_0 + I_1$.

$$I_0 + I_1 = \int \frac{(1+x)e^{\arctan x}}{\sqrt{1+x^2}} dx$$

Substitute $\arctan x = t$, $x = \tan(t)$ and $dx = \sec^2(t)dt$

$$\int \frac{(1+x)e^{\arctan x}}{\sqrt{1+x^2}} dx = \int \frac{(1+\tan(t))e^t}{\sqrt{1+\tan^2(t)}} \sec^2(t)dt = \int (1+\tan t)e^t \sec t dt$$

. Note, that $(e^t \sec t)' = e^t \sec t \tan t + e^t \sec t = (1+\tan t)e^t \sec t$. So,

$$\int (1+\tan t)e^t \sec t dt = e^t \sec t = e^{\arctan x} \sqrt{1+\tan^2 t} = e^{\arctan x} \sqrt{1+x^2}$$

. Thus,

$$I_0 + I_1 = e^{\arctan x} \sqrt{1+x^2}$$

By , our identity we have , .

$$I_2 = \frac{(x-1)\sqrt{1+x^2}e^{\arctan x}}{2}$$

Problem 1.2 (U661, July, Issue 3, 2024). Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(xy) = f(x)f(y+1) - (2y+1)f(x),$$

for all $x, y \in \mathbb{R}$

Solution 1.2 (U661). Put $x = 0$ and $y = 0$, we get $f(0)(f(1) - 2) = 0$.

Case 1 $f(0) = 0$

Set $y = 0$ to get $f(x)(f(1) - 1) = 0$. So, $f(x) = 0$ is one solution. The other case is $f(1) = 1$. Thus, $f(0) = 0$ and $f(1) = 1$.

Now plug in $x = 1$ to get:

$$f(y) = f(1)f(y+1) - (2y+1)f(1) = f(y+1) - (2y+1)$$

So $f(xy) = f(x)(f(y+1) - (2y+1)) = f(x)f(y)$. Hence, the function is multiplicative.

Set $x = y = 1$ in 1.2, we have that $f(2) = f(1)f(2) + (2 \cdot 1 + 1) = 4$.

Claim -: f is an even function and $f(x) \geq 0$ for every $x \in \mathbb{R}$.

Set $y = -1$ in the original equation to get $f(-x) = f(x)(f(0) - (-2 + 1)) = f(x)$. Thus, f is an even function.

Now, we will prove that $f(x) \geq 0$ for all $x \in \mathbb{R}$. As f is even, we only need to prove that $f(x) \geq 0$ for $x \geq 0$.

$$f(x) = f(\sqrt{x}\sqrt{x}) = f(\sqrt{x})f(\sqrt{x}) = (f(\sqrt{x}))^2$$

So, clearly $f(x) \geq 0$ for $x \geq 0$.

Now, as f is positive on the ray $x \geq 0$, by characterization of such functional equation, through positivity of the function on ray $x \geq 0$ as documented in hyperlinked [IMO Compendium](#), the Cauchy functional equation solution is $f(x) = x^k$ for some $k \in \mathbb{N}$.

Since $f(2) = 4$, we have $k = 2$. Thus, $f(x) = x^2$ is the solution of the functional equation. **Case 2 : $f(0) \neq 0$**

Put $x = 0$ and $y = 0$, we get $f(0)(f(1) - 2) = 0$. So $f(1) = 2$. Now plug in $x = 1$ to get:

$$f(y) = f(1)f(y+1) - (2y+1)f(1) = 2(f(y+1) - (2y+1))$$

$$f(y+1) - (2y+1) = \frac{f(y)}{2}$$

$$f(y+1) = \frac{f(y)}{2} + (2y+1)$$

Define $a_n = f(n)$, we have the recursion $a_{n+1} = \frac{a_n}{2} + 2n + 1$ with $a_1 = 2$. Now note that $a_2 = 2/2 + 2 * 1 + 1 = 4$, $a_3 = 4/2 + 2 * 2 + 1 = 7$, $a_4 = 7/2 + 2 * 3 + 1 = 21/2$, $a_5 = 21/4 + 2 * 4 + 1 = 21/4 + 9 = 57/4$. Through these values we observe that

$$a_n = f(n) = 4n + 2^{3-n} - 6$$

We will prove that this hold true by induction for $n \geq 2$.

For the base case, $a_2 = 4 = 4 * 2 + 2^{3-2} - 6$.

Induction hypothesis is

$$a_n = 4n + 2^{3-n} - 6$$

We have to prove that

$$a_{n+1} = 4(n+1) + 2^{3-(n+1)} - 6 = 4n + 2^{2-n} - 2$$

Now note that

$$a_{n+1} = \frac{a_n}{2} + 2n + 1$$

$$\text{By induction hypothesis, } \frac{a_n}{2} + 2n + 1 = \frac{4n + 2^{3-n} - 6}{2} + 2n + 1$$

$$\frac{4n + 2^{3-n} - 6}{2} + 2n + 1 = 2n + 2^{2-n} - 3 + 2n + 1 = 4n + 2^{2-n} - 2$$

Thus, $a_n = f(n) = 4n + 2^{3-n} - 6$. But, Note that a_n does not satisfies the original equation : To check whether $f(x) = 4x + 2^{3-x} - 6$ satisfies the functional equation $f(xy) = f(x)f(y+1) - (2y+1)f(x)$, we need to verify this equation for all x and y .

Given:

$$f(x) = 4x + 2^{3-x} - 6$$

First, calculate $f(xy)$:

$$f(xy) = 4(xy) + 2^{3-xy} - 6$$

Next, calculate $f(x)f(y+1)$:

$$\begin{aligned} f(x) &= 4x + 2^{3-x} - 6 \\ f(y+1) &= 4(y+1) + 2^{3-(y+1)} - 6 = 4y + 4 + 2^{2-y} - 6 = 4y - 2 + 2^{2-y} \\ f(x)f(y+1) &= (4x + 2^{3-x} - 6)(4y - 2 + 2^{2-y}) \end{aligned}$$

Expand $f(x)f(y+1)$:

$$\begin{aligned} f(x)f(y+1) &= (4x + 2^{3-x} - 6)(4y - 2 + 2^{2-y}) \\ &= 4x(4y - 2 + 2^{2-y}) + 2^{3-x}(4y - 2 + 2^{2-y}) - 6(4y - 2 + 2^{2-y}) \\ &= 16xy - 8x + 4x \cdot 2^{2-y} + 4y \cdot 2^{3-x} - 2 \cdot 2^{3-x} + 2^{3-x} \cdot 2^{2-y} - 24y + 12 - 6 \cdot 2^{2-y} \end{aligned}$$

Now, calculate $(2y+1)f(x)$:

$$\begin{aligned} (2y+1)f(x) &= (2y+1)(4x + 2^{3-x} - 6) \\ &= (2y+1)(4x) + (2y+1)(2^{3-x}) - (2y+1)6 \\ &= 8xy + 4x + 2y \cdot 2^{3-x} + 2^{3-x} - 12y - 6 \end{aligned}$$

Combine the terms for $f(x)f(y+1) - (2y+1)f(x)$:

$$\begin{aligned} f(x)f(y+1) - (2y+1)f(x) &= (16xy - 8x + 4x \cdot 2^{2-y} + 4y \cdot 2^{3-x} - 2 \cdot 2^{3-x} + 2^{3-x} \cdot 2^{2-y} - 24y + 12 - 6 \cdot 2^{2-y}) \\ &\quad - (8xy + 4x + 2y \cdot 2^{3-x} + 2^{3-x} - 12y - 6) \\ &= 16xy - 8x + 4x \cdot 2^{2-y} + 4y \cdot 2^{3-x} - 2 \cdot 2^{3-x} + 2^{3-x} \cdot 2^{2-y} - 24y + 12 - 6 \cdot 2^{2-y} - 8xy - 4x - 2y \cdot 2^{3-x} - 2^{3-x} + 12y + 6 \\ &= 8xy - 12x + 4x \cdot 2^{2-y} + 2 \cdot 2^{3-x} \cdot 2^{2-y} - 12y + 18 - 6 \cdot 2^{2-y} - 2^{3-x} \end{aligned}$$

Now, compare $f(xy)$ and $f(x)f(y+1) - (2y+1)f(x)$:

$$f(xy) = 4xy + 2^{3-xy} - 6$$

$$f(x)f(y+1) - (2y+1)f(x) = 8xy - 12x + 4x \cdot 2^{2-y} + 2 \cdot 2^{3-x} \cdot 2^{2-y} - 12y + 18 - 6 \cdot 2^{2-y} - 2^{3-x}$$

Clearly, $f(xy)$ does not equal $f(x)f(y+1) - (2y+1)f(x)$ in general. Hence, $f(x) = 4x + 2^{3-x} - 6$ does not satisfy the functional equation $f(xy) = f(x)f(y+1) - (2y+1)f(x)$.

Problem 1.3 (U 668, September, Issue 4, 2024). Evaluate

$$\int_0^1 \frac{-5x^2 + 5x + 4}{x^5 + x + 1} dx$$

Solution 1.3 (U 668). Note that

$$\int_0^1 \frac{-5x^2 + 5x + 4}{x^5 + x + 1} dx = \int_0^1 \frac{3x + 4}{x^2 + x + 1} dx + \int_0^1 \frac{2x - 3x^2}{x^3 - x^2 + 1} dx$$

We will first evaluate

$$\int_0^1 \frac{2x - 3x^2}{x^3 - x^2 + 1} dx$$

Substitute $t = x^3 - x^2 + 1$. $dt = -(2x + 3x^2)dx$. For $x = 0$, $t = 1$. Also for $x = 1$, $t = 1$. So it is

$$\int_1^1 -\frac{dt}{t} = 0$$

Now we will evaluate

$$\int_0^1 \frac{3x + 4}{x^2 + x + 1} dx$$

We can rewrite the integral as

$$\int_0^1 \frac{3(2x+1)}{2(x^2+x+1)} dx + \int_0^1 \frac{5}{2(x^2+x+1)} dx$$

$$\frac{3}{2} \left(\int_0^1 \frac{2x+1}{x^2+x+1} dx \right) + \frac{5}{2} \left(\int_0^1 \frac{1}{x^2+x+1} dx \right)$$

We will now individual find both of the integrals. For the first integral substitute $t = x^2 + x + 1$. We have $dt = (2x+1)dx$ For $x = 0$, $t = 1$ and for $x = 1$, $t = 3$

$$\int_0^1 \frac{2x+1}{x^2+x+1} dx = \int_1^3 \frac{dt}{t} = \ln(3) - \ln(1) = \ln(3)$$

Now we will find

$$\int_0^1 \frac{1}{x^2+x+1} dx$$

To evaluate the integral

$$\int_0^1 \frac{1}{x^2+x+1} dx,$$

We first complete the square to write the integral in the standard form.

Rewrite $x^2 + x + 1$ as follows:

$$x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 - \frac{1}{4} + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}.$$

Now we substitute $u = x + \frac{1}{2}$. Then we get $du = dx$, and when $x = 0$, $u = \frac{1}{2}$; when $x = 1$, $u = \frac{3}{2}$.

Rewrite the integral in terms of u :

$$\int_0^1 \frac{1}{x^2+x+1} dx = \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{u^2 + \frac{3}{4}} du.$$

Note that this integral can be solved using the standard integral formula:

$$\int \frac{1}{a^2 + u^2} du = \frac{1}{a} \arctan\left(\frac{u}{a}\right).$$

Here, $a^2 = \frac{3}{4}$, so $a = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$.

$$\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{u^2 + \frac{3}{4}} du = \frac{1}{\sqrt{3}/2} \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2 + u^2} du.$$

$$= \frac{2}{\sqrt{3}} \left[\arctan\left(\frac{2u}{\sqrt{3}}\right) \right]_{\frac{1}{2}}^{\frac{3}{2}}.$$

So,

$$\int_0^1 \frac{1}{x^2+x+1} dx = \frac{2}{\sqrt{3}} \left[\arctan\left(\frac{3}{\sqrt{3}}\right) - \arctan\left(\frac{1}{\sqrt{3}}\right) \right].$$

Using the fact that:

$$\arctan\left(\frac{3}{\sqrt{3}}\right) = \frac{\pi}{3} \quad \text{and} \quad \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6},$$

we have:

$$\int_0^1 \frac{1}{x^2 + x + 1} dx = \frac{2}{\sqrt{3}} \left(\frac{\pi}{3} - \frac{\pi}{6} \right) = \frac{2}{\sqrt{3}} \cdot \frac{\pi}{6} = \frac{\pi}{3\sqrt{3}}.$$

Thus, the value of the original integral is

$$\boxed{\frac{5\pi}{6\sqrt{3}} + \frac{3\ln(3)}{2}}$$

2. Olympiad Problems

Problem 2.1 (O 667, September, Issue 4, 2024). Prove that for any positive real numbers a, b, c

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + \frac{9(a+b)(b+c)(c+a)}{(a+b+c)^3} \geq \frac{25}{6}$$

Solution 2.1 (O 667). Subtract $\frac{3}{2}$ from both sides of the equation to get

$$\sum_{cyc} \frac{a-b+a-c}{2(b+c)} + \frac{9(a+b)(b+c)(c+a)}{(a+b+c)^3} \geq \frac{8}{3}$$

We can write

$$\sum_{cyc} \frac{a-b+a-c}{2(b+c)} = \sum_{cyc} \frac{a-b}{2(b+c)} + \frac{b-a}{2(c+a)} = \sum_{cyc} \frac{(a-b)^2}{2(b+c)(c+a)}$$

By AM - GM

$$\frac{(a-b)^2}{2(b+c)(c+a)} + \frac{3(a+b)(b+c)(c+a)}{(a+b+c)^3} \geq \sqrt{\frac{6(a+b)(a-b)^2}{(a+b+c)^3}}$$

To prove that

$$\sum_{cyc} (a-b) \sqrt{6(a+b)} \geq \frac{8\sqrt{(a+b+c)^3}}{3}$$

Problem 2.2 (O 668, September, Issue 4, 2024). How many ordered 4-tuples (a, b, c, d) of pairwise distinct numbers $a, b, c, d \in \{1, 2, \dots, 17\}$ are there such that $a - b + c - d$ is divisible by 17.

Solution 2.2 (O 668). Note that $a - b + c - d$ has 3 possible values that are 0, -17, 17.

Case- 1) $a - b + c - d = -17$. So we have

$$a + c = b + d - 17$$

. Now clearly $b + d \geq 20$. Denote $b = a + k_1$, $c = b + k_2$ and $d = c + k_3$. $k_1 \neq 0, k_2 \neq 0, k_3 \neq 0$. $k_1 + k_3 = 17$ where $k_1 \neq 0$ and $k_3 \neq 0$. 16 solution. Corresponding of $k_1 = 1$, $a = 16$ value of a possible. In total $16(1 + 2 + 3 + 4 \dots 16) = 2176$

Case 2) $a + c = b + d$. $k_1 = -k_3$. 32 different possible values of k_1 . And for a it is between 1 to 17. $32(1 + 2 \dots 17) = 4896$ Case 3) Same as case 1 In total answer is 9248.

3. Senior Problems

Problem 3.1 (S661). [July Issue 3, 2024] Find all n for which there is a positive integer k such that

$$(n-1)! + 1 = (k-30)^2 \text{ and } (n+1)! + 1 = (k+30)^2$$

Solution 3.1 (S661). Subtracting the two given equations, we get

$$\begin{aligned} (n+1)! + 1 - (n-1)! - 1 &= (k+30)^2 - (k-30)^2 \\ (n-1)!(n^2 + n - 1) &= 120k \\ \frac{(n-1)!(n^2 + n - 1)}{120} &= k \end{aligned}$$

Substitute this value of k in the equation $(n+1)! + 1 = (k+30)^2$ we get

$$\left(\frac{(n-1)!(n^2 + n - 1)}{120} + 30\right)^2 = (n+1)! + 1$$

Claim : For $n \geq 10$, $\left(\frac{(n-1)!(n^2 + n - 1)}{120} + 30\right)^2 > (n+1)! + 1$. Note that for $n \geq 10$,

$$\left(\frac{(n-1)!(n^2 + n - 1)}{120} + 30\right)^2 > ((n-1)!)^2$$

So, we will now prove that $((n-1)!)^2 > (n+1)! + 1$. Proof by induction. It is easy to verify it for $n = 10$. induction hypothesis is

$$((n-1)!)^2 > (n+1)! + 1$$

. We have to prove that

$$(n!)^2 > (n+2)! + 1$$

Note that $(n!)^2 = (n(n-1)!)^2 = n^2((n-1)!)^2 > n^2((n+1)! + 1)$ (By induction hypothesis).

Now, $n^2(n+1)! + n^2 > (n+2)(n+1)! + n^2 > (n+2)! + 1$ (For $n \geq 10$, $n^2 > n+2$).

Thus we are done and $n \leq 9$. Checking the values from 1 to 9, we find that only $n = 6$ and $k = 41$ satisfies both the equation.

Problem 3.2 (S663, July, Issue 3, 2024). Let ABC be a triangle with $\angle A = \angle B + 60^\circ$. Prove that

$$(\cos 3A + i \sin 3B)^4 + (\cos 3B + i \sin 3A)^4 = 2(\cos 3C + i \sin 3C)^2.$$

Solution 3.2 (S663). Now, $\therefore 3\angle A = 180^\circ + 3\angle B$. Using trigonometric identities, we have:

$$\cos 3A = \cos(3B + 180^\circ) = -\cos 3B,$$

$$\sin 3A = \sin(3B + 180^\circ) = -\sin 3B.$$

Now, consider the expression:

$$(\cos 3A + i \sin 3B)^4 + (\cos 3B + i \sin 3A)^4 = (-\cos 3B + i \sin 3B)^4 + (\cos 3B - i \sin 3B)^4.$$

$$(-\cos 3B + i \sin 3B)^4 + (\cos 3B - i \sin 3B)^4 = 2(\cos 3B - i \sin 3B)^4 = 2(e^{i(-3B)})^4 = 2e^{i(-12B)}$$

So,

$$(\cos 3A + i \sin 3B)^4 + (\cos 3B + i \sin 3A)^4 = 2e^{i(-12B)}$$

Now, $\angle C = 180^\circ - \angle A - \angle B = 120^\circ - 2\angle B$. Using trigonometric identities, we have:

$$\cos 3C = \cos(360^\circ - 6\angle B) = \cos 6B,$$

$$\sin 3C = \sin(360^\circ - 6\angle B) = -\sin 6B.$$

So,

$$2(\cos 3C + i \sin 3C)^2 = 2(\cos 6B - i \sin 6B)^2 = 2(e^{i(-6B)})^2 = 2e^{i(-12B)}$$

Hence,

$$(\cos 3A + i \sin 3B)^4 + (\cos 3B + i \sin 3A)^4 = 2(\cos 3C + i \sin 3C)^2 = 2e^{i(-12B)}$$

Problem 3.3 (S666, July, Issue 3, 2024). Let $0 \leq a_1 \leq 1 \leq a_2 \leq \dots \leq a_n$ such that $a_1 + a_2 + \dots + a_n = n$. Prove that

$$a_1a_2 + a_2a_3 + \dots + a_na_1 \leq n$$

Solution 3.3 (S666). We are given that $0 \leq a_1 \leq 1 \leq a_2 \leq \dots \leq a_n$ and the sum of all a_i is n . The rearrangement inequality states that for two sequences $x_1 \leq x_2 \leq \dots \leq x_n$ and $y_1 \leq y_2 \leq \dots \leq y_n$,

$$x_1y_1 + x_2y_2 + \dots + x_ny_n \leq x_1y_{\sigma(1)} + x_2y_{\sigma(2)} + \dots + x_ny_{\sigma(n)}$$

for any permutation σ .

Let $x_i = a_i$ and $y_i = a_{i+1}$ (with $a_{n+1} = a_1$). Since $a_1 \leq a_2 \leq \dots \leq a_n$, we can apply the rearrangement inequality:

$$a_1a_2 + a_2a_3 + \dots + a_na_1 \leq a_1a_1 + a_2a_2 + \dots + a_na_n$$

Since $a_1 \leq 1$ and $a_2, a_3, \dots, a_n \geq 1$, we have:

$$a_1a_2 + a_2a_3 + \dots + a_na_1 \leq a_1^2 + a_2^2 + \dots + a_n^2$$

Using the fact that $a_1 + a_2 + \dots + a_n = n$, we apply the Cauchy-Schwarz inequality:

$$(a_1^2 + a_2^2 + \dots + a_n^2)(1^2 + 1^2 + \dots + 1^2) \geq (a_1 + a_2 + \dots + a_n)^2$$

Simplifying, we get:

$$(a_1^2 + a_2^2 + \dots + a_n^2) \cdot n \geq n^2$$

$$a_1^2 + a_2^2 + \dots + a_n^2 \geq n$$

Combining the results, we have:

$$a_1a_2 + a_2a_3 + \dots + a_na_1 \leq a_1^2 + a_2^2 + \dots + a_n^2 \leq n$$

Therefore, we have proven that $a_1a_2 + a_2a_3 + \dots + a_na_1 \leq n$.

This completes the formal proof.

Problem 3.4 (S 669, September, Issue 4, 2024). Let a, b, c be positive real numbers such that $ab + bc + ca = 1$. Prove that

$$\sum_{cyc} bc\sqrt{2a+b+c} \geq \sqrt{\frac{9abc}{4} + \frac{3(a+b+c)}{4}} + \frac{1}{a+b+c}$$

Solution 3.4 (S 669). By cauchy schwartz we have

$$\left(\sum_{cyc} bc\sqrt{2a+b+c} \right)^2 \geq \left(\sum_{cyc} bc \right) \left(\sum_{cyc} bc(2a+b+c) \right)$$

Note that $\left(\sum_{cyc} bc \right) = 1$ and $\left(\sum_{cyc} bc(2a+b+c) \right) = 3abc + (a+b+c)(ab+bc+ca) = 3abc + a + b + c$.

So

$$\sum_{cyc} bc\sqrt{2a+b+c} \geq \sqrt{3abc + a + b + c}$$

It is enough to prove that

$$\begin{aligned} 3abc + a + b + c &\geq \frac{9abc}{4} + \frac{3(a+b+c)}{4} + \frac{1}{a+b+c} \\ \frac{3abc}{4} + \frac{a+b+c}{4} &\geq \frac{1}{a+b+c} \\ (3abc + a + b + c)(a+b+c) &\geq 4 \end{aligned}$$

Note that $(a+b+c)^2 = a^2 + b^2 + c^2 + 2 \geq ab + bc + ca + 2 = 3$. It is enough to prove that

$$3abc(a+b+c) \geq 1$$

But the last inequality proof is obvious.

Problem 3.5 (S 671, September, Issue 4, 2024). Let ABC be a triangle with circumradius R and let Ω be its Brocard point such that $\angle\Omega AB = \angle\Omega BC = \angle\Omega CA = \alpha$. Prove that if $\cot \alpha = 5 \cot A$ and $BC = a$ then

$$\cos(A) \geq \frac{1}{3} a \leq 3R\sqrt{\frac{3}{5}}$$

Solution 3.5 (S 671). Now $\cot \alpha = \cot A + \cot B + \cot C$. So we get $4 \cot A = \cot B + \cot C = \frac{\sin(B+C)}{\sin B \sin C}$. Thus,

$$\begin{aligned} 4 \cos A \sin B \sin C &= \sin^2 A \\ 2 \cos A (\cos B - C + \cos A) &= \sin^2 A \\ 3 \cos^2 A + 2 \cos A \cos B - C &= 1 \\ 3 \cos^2 A &= 1 + \cos 3C + \cos 3B \end{aligned}$$

Clearly $\cos A \geq \frac{1}{3}$

4. Junior Problems

Problem 4.1 (J661, July, Issue 3, 2024). Find all positive integers n for which

$$\frac{(n^2+1)! + (n^2)! + (n^2-1)!}{(n+1)!} = (n+3)^2.$$

Solution 4.1 (J661). Note that $n = 2$ clearly satisfies the equation.

Claim: For $n \geq 3$, we have

$$\frac{(n^2+1)! + (n^2)! + (n^2-1)!}{(n+1)!} > (n+3)^2$$

Now,

$$\frac{(n^2+1)! + (n^2)! + (n^2-1)!}{(n+1)!} > \frac{(n^2+1)!}{(n+1)!} = (n^2+1)(n^2)(n^2-1)(n^2-2) \cdots (n+2)$$

If we show that $(n^2+1)n^2 \geq (n+3)^2$ for $n \geq 3$, then we are done. This is equivalent to showing that $n^4 - 6n - 9 > 0$ for $n \geq 3$.

As $f(3) > 0$, it is sufficient to prove that $f(x) = x^4 - 6x - 9$ is increasing for $x \geq 3$. We need to show that $f(x_2) > f(x_1)$ whenever $x_2 > x_1 \geq 3$.

Consider $f(x) = x^4 - 6x - 9$.

$$\begin{aligned}
f(x_2) - f(x_1) &= (x_2^4 - 6x_2 - 9) - (x_1^4 - 6x_1 - 9) \\
&= x_2^4 - x_1^4 - 6(x_2 - x_1) \\
&= (x_2 - x_1)(x_2^3 + x_2^2x_1 + x_2x_1^2 + x_1^3) - 6(x_2 - x_1) \\
&= (x_2 - x_1)(x_2^3 + x_1^3 + x_2^2x_1 + x_2x_1^2 - 6)
\end{aligned}$$

Since $x_2 > x_1$, $x_2 - x_1 > 0$. We also need to show that $x_2^3 + x_1^3 + x_2^2x_1 + x_2x_1^2 - 6$ is positive for $x_2 > x_1 \geq 3$. Therefore,

$$f(x_2) - f(x_1) > 0$$

which means $f(x_2) > f(x_1)$. Hence, by the definition of an increasing function, $f(x) = x^4 - 6x - 9$ is increasing for $x \geq 3$.

Problem 4.2 (J663, July, Issue 3, 2024). Let $f : (0, \infty) \rightarrow \mathbb{R}$ be a function such that

$$\frac{x}{y}f\left(\frac{y}{x}\right) + \left(\frac{x}{y} - \frac{y}{x}\right)f\left(\frac{x}{y} + \frac{y}{x}\right) + \frac{y}{x}f\left(\frac{x}{y}\right) = \frac{x^3}{y^3} - \frac{y^3}{x^3}$$

for all positive real numbers x and y . Evaluate $f(2024)$.

Solution 4.2 (J663). let $u = \frac{x}{y}$, then the equation is

$$(1) \quad uf\left(\frac{1}{u}\right) + \left(u - \frac{1}{u}\right)f\left(u + \frac{1}{u}\right) + \frac{1}{u}f(u) = u^3 - \frac{1}{u^3}$$

Replace u by $\frac{1}{u}$ in the original equation we get

$$(2) \quad \frac{1}{u}f(u) + \left(\frac{1}{u} - u\right)f\left(\frac{1}{u} + u\right) + uf\left(\frac{1}{u}\right) = \frac{1}{u^3} - u^3$$

Adding equation 1 and 2 we get ,

$$2\frac{1}{u}f(u) + 2uf\left(\frac{1}{u}\right) = 0$$

$$(3) \quad \frac{1}{u}f(u) + uf\left(\frac{1}{u}\right) = 0$$

Now , using equation 1 and 3 we get

$$\begin{aligned}
\left(u - \frac{1}{u}\right)f\left(\frac{1}{u} + u\right) &= u^3 - \frac{1}{u^3} \\
f\left(\frac{1}{u} + u\right) &= \frac{u^3 - \frac{1}{u^3}}{u - \frac{1}{u}} \\
f\left(\frac{1}{u} + u\right) &= u^2 + \frac{1}{u^2} + 1
\end{aligned}$$

let $t = \frac{1}{u} + u$, then we have $f(t) = t^2 - 1$. Thus , $f(x) = x^2 - 1$ for $x \geq 2$. (As $t = \frac{1}{u} + u$, has solution when $t \geq 2$)

Problem 4.3 (J664, July, Issue 3, 2024). Let a, b, c be non-negative real numbers such that $ab + bc + ca \geq 3$. Prove that

$$\frac{1}{a^2 + 2} + \frac{1}{b^2 + 2} + \frac{1}{c^2 + 2} \leq 1.$$

Solution 4.3 (J664). This is equivalent to proving that

$$\begin{aligned}\frac{2}{a^2+2} + \frac{2}{b^2+2} + \frac{2}{c^2+2} &\leq 2. \\ \frac{2}{a^2+2} - 1 + \frac{2}{b^2+2} - 1 + \frac{2}{c^2+2} - 1 &\leq 2 - 3 = -1 \\ \frac{a^2}{a^2+2} + \frac{b^2}{b^2+2} + \frac{c^2}{c^2+2} &\geq 1\end{aligned}$$

By, titu lemma we have

$$\frac{a^2}{a^2+2} + \frac{b^2}{b^2+2} + \frac{c^2}{c^2+2} \geq \frac{(a+b+c)^2}{a^2+b^2+c^2+6}$$

Now, $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab+bc+ca) \geq a^2 + b^2 + c^2 + 6$ (As $ab+bc+ca \geq 3$) So we are done .

Problem 4.4 (J666, July, Issue 3, 2024). Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(xy) = f(x)f(y+1) + f(x-1)f(y)$$

Solution 4.4 (J666). Substitute $x = 0$ into the given functional equation:

$$f(0 \cdot y) = f(0)f(y+1) + f(-1)f(y)$$

This simplifies to:

$$f(0) = f(0)f(y+1) + f(-1)f(y)$$

Let's denote $f(0) = c$ and $f(-1) = d$, so we have:

$$c = cf(y+1) + df(y)$$

Considering $c = f(0)$: Let's analyze the cases for $f(0)$: Case 1: $f(0) = 0$ If $c = 0$, we get:

$$0 = 0 \cdot f(y+1) + df(y) \implies df(y) = 0 \implies f(y) = 0$$

In this case, the solution is:

$$f(x) = 0 \quad \text{for all } x \in \mathbb{R}$$

Case 2: $f(0) \neq 0$ If $c \neq 0$, we can divide by c in the equation:

$$1 = f(y+1) + \frac{d}{c}f(y)$$

Let $e = \frac{d}{c}$, then:

$$f(y+1) = 1 - ef(y)$$

Substitute $y = 0$ in the original functional equation:

$$f(x \cdot 0) = f(x)f(0+1) + f(x-1)f(0)$$

This simplifies to:

$$f(0) = f(x)f(1) + f(x-1)f(0)$$

Using $f(0) = c$, we get:

$$c = f(x)f(1) + cf(x-1)$$

As $c \neq 0$, we can divide by c :

$$1 = f(x)\frac{f(1)}{c} + f(x-1)$$

Let $m = \frac{f(1)}{c}$, then:

$$1 = mf(x) + f(x-1)$$

Combining Relationships We now have two relationships:

$$f(y+1) + ef(y) = 1$$

$$1 = mf(x) + f(x-1)$$

These are consistent if $f(x)$ is a constant function. Let $f(x) = k$ for all $x \in \mathbb{R}$. Substituting $f(x) = k$ in the original functional equation, we get:

$$k = k \cdot k + k \cdot k \implies k = 2k^2 \implies 2k^2 - k = 0 \implies k(2k - 1) = 0$$

Thus, $k = 0$ or $k = \frac{1}{2}$.

Problem 4.5 (J 667, September, Issue 4, 2024). Prove that the sum of squares of two twin primes can be written as the sum of squares of three positive integers.

Solution 4.5 (J 667). Clearly the twin primes are of the form $(6k-1, 6k+1)$ for some integer k . Sum of squares is

$$(6k+1)^2 + (6k-1)^2 = 72k^2 + 2 = (8k)^2 + (2k+1)^2 + (2k-1)^2$$

Problem 4.6 (J 669, September, Issue 4, 2024). Let a, b, c be positive real numbers. Prove that

$$\frac{a+b}{a^2-ab+b^2} + \frac{b+c}{b^2-bc+c^2} + \frac{c+a}{c^2-ca+a^2} + \frac{ab+bc+ca}{abc} \geq \frac{3(a+b+c)^2}{a^3+b^3+c^3}$$

Solution 4.6 (J 669). Cauchy Schwartz states that

$$\left(\sum_{i=1}^n a_i b_i \right)^2 \leq \left(\sum_{i=1}^n a_i^2 \right) \left(\sum_{i=1}^n b_i^2 \right).$$

Take $a_1 = \sqrt{\frac{a+b}{a^2-ab+b^2}}$, $a_2 = \sqrt{\frac{b+c}{b^2-bc+c^2}}$, etc. and $b_1 = \sqrt{(a^2-ab+b^2)(a+b)}$,

$b_2 = \sqrt{(b^2-bc+c^2)(b+c)}$, etc. Now $a_1 b_1 = a+b$, $a_2 b_2 = b+c$ and $a_3 b_3 = c+a$. So by Cauchy Schwartz we get

$$\begin{aligned} (4) \quad & \left(\frac{a+b}{a^2-ab+b^2} + \frac{b+c}{b^2-bc+c^2} + \frac{c+a}{c^2-ca+a^2} \right) \\ & \times [(a^2-ab+b^2)(a+b) + (b^2-bc+c^2)(b+c) + (c^2-ac+a^2)(a+c)] \\ & \geq (a+b+b+c+c+a)^2 \\ & = (2a+2b+2c)^2 = 4(a+b+c)^2 \end{aligned}$$

Simplifying we get that ,

$$\frac{a+b}{a^2-ab+b^2} + \frac{b+c}{b^2-bc+c^2} + \frac{c+a}{c^2-ca+a^2} \geq \frac{2(a+b+c)^2}{a^3+b^3+c^3}$$

Thus to prove that

$$\frac{ab+bc+ca}{abc} \geq \frac{(a+b+c)^2}{a^3+b^3+c^3}$$

To prove that

$$\frac{a^2}{a^3} + \frac{b^2}{b^3} + \frac{c^2}{c^3} \geq \frac{(a+b+c)^2}{a^3+b^3+c^3}$$

The above inequality is obvious by Titus's lemma.

Problem 4.7 (J 671, September, Issue 4, 2024). Let $a > 1$ be a real number. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x+y) = f(x)f(y) - a \sin x \sin y,$$

for all $x, y \in \mathbb{R}$.

Solution 4.7 (J 671). The potential solutions are of form Based on our calculations, let's analyze the results for the function form $f(x) = A \cos(x) + B \sin(x)$, where A and B are constants to be determined.

Let's compare the left-hand side (LHS) and right-hand side (RHS) of our functional equation:

$$\text{LHS: } f(x+y) = A \cos(x+y) + B \sin(x+y)$$

$$f(x)f(y) = A \sin(x) \sin(y) = A^2 \cos(x) \cos(y) + AB \sin(x) \cos(y) + AB \sin(y) \cos(x) + B^2 \sin(x) \sin(y)$$

For these to be equal for all x and y , we need to match the coefficients:

- Coefficient of $\cos(x) \cos(y)$: $A = A^2$
- Coefficient of $\sin(x) \cos(y)$ and $\cos(x) \sin(y)$: $B = AB$
- Coefficient of $\sin(x) \sin(y)$: $-A = B^2 - a$

From equation 1, we can deduce that either $A = 0$ or $A = 1$.

Case 1: If $A = 0$

- Note that $B = 0$ (since $B = AB = 0 \cdot B = 0$)
- But this contradicts the last written equation as $-0 \neq B^2 - a$ ($a > 1$)

Case 2: If $A = 1$

- From equation 2, we get $B = B$, which is always true.
- From equation 3, we get $-1 = B^2 - a$, or $B^2 = a - 1$

The calculation confirms that the function we hypothesized indeed satisfies the given functional equation for $a > 1$. $f(x) = \cos(x) \pm \sqrt{a-1} \sin(x)$ Therefore, the solution to the functional equation

$$f(x+y) = f(x)f(y) - a \sin x \sin y$$

is

$$f(x) = \cos(x) \pm \sqrt{a-1} \sin(x)$$

for $a > 1$.

Problem 4.8 (J 672, September, Issue 4, 2024). Let $ABCD$ be a convex quadrilateral such that

$$(AB - BC)^2 + (CD - DA)^2 = 2(AB \cdot AD - AC \cdot BD + BC \cdot CD)$$

. Prove that the statements

- $ABCD$ is cyclic.
- $ABCD$ is circumscribable.

Solution 4.8 (J 672). We will first prove that (a) implies (b). Now denote $AB = a, BC = b, CD = c, DA = d, AC = e, BD = f$. As $ABCD$ is cyclic, we have $AB \cdot CD + BC \cdot AD =$

$AC \cdot BD$. So , $ac + bd = ef$.

$$(a - b)^2 + (c - d)^2 = 2(a \cdot d - e \cdot f + b \cdot c)$$

$$a^2 + b^2 - 2ab + c^2 + d^2 - 2cd = 2(ad - ef + bc)$$

$$a^2 + b^2 + c^2 + d^2 = 2(ab + cd + ad + bc - ac - bd) \quad (\text{As } ef = ac + bd)$$

$$(a^2 + c^2 + 2ac) + (b^2 + d^2 + 2bd) = 2(ab + cd + ad + bc)$$

$$(a + c)^2 + (b + d)^2 = 2(a + c)(b + d)$$

$$(a + c - b - d)^2 = 0$$

So $a + c = b + d$, i.e $AB + CD = AD + BC$. By pitot theorem , $ABCD$ is circumscribable.

Now we will prove that (b) implies (a). $ABCD$ is circumscribable so by pitot theorem , $AB + CD = AD + BC$. i.e $a + c = b + d$. So we have

$$(a + c - b - d)^2 = 0$$

$$a^2 + b^2 + c^2 + d^2 = 2(ab + cd + ad + bc - ac - bd)$$

Now the given info is

$$(a - b)^2 + (c - d)^2 = 2(a \cdot d - e \cdot f + b \cdot c)$$

$$a^2 + b^2 - 2ab + c^2 + d^2 - 2cd = 2(ad - ef + bc)$$

$$a^2 + b^2 + c^2 + d^2 = 2(ab + cd + ad + bc - ef)$$

Equating both values of $a^2 + b^2 + c^2 + d^2$ we get ,

$$2(ab + cd + ad + bc - ef) = 2(ab + cd + ad + bc - ac - bd)$$

$$ef = ac + bd$$

Thus, $ac + bd = ef$. In other words, $AB \cdot CD + BC \cdot AD = AC \cdot BD$. by Ptolemy $ABCD$ is cyclic.